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Mathematical Reviews

Vol. 21, No. 3 March, 1960 Reviews 1262–1929

LOGIC AND FOUNDATIONS

1262:

Kotarbińska, Janina. The concept of sign. Studia Logica 6 (1957), 57–143. (Polish. Russian and English summaries)

After a critical survey of various leading characterizations of the general notion of sign, the author presents her own. Various aspects of the s.c. objective (semantical) binary "sign-relation" (A is the sign of an object B) are considered and also the ternary s.c. subjective (pragmatical) "sign-relation" (A is the sign, for a person O, of an object B). The author defines the latter by the former, i.e. by the condition that O believes the former to be the case. This submitting of the pragmatical to the semantical aspect constitutes the author's s.c. objectivist standpoint which seems to be quite natural in logical analysis. The subtle discussion belongs rather to general linguistics or to gnoseology than to foundations of mathematics or to mathematical logic. *L. Rieger* (Prague)

1263:

Kubiński, Tadeusz. Vague terms. Studia Logica 7 (1958), 115–179. (Polish. Russian and English summaries)

This is an attempt at a formally correct theory of incorrect (vague) terms. The reviewer has been unable to understand to what extent this attempt has been successful, for the (almost trivial) formal apparatus introduced seems to have little to do with the problems of the real use of vague terms. *L. Rieger* (Prague)

1264:

Reichbach, Juliusz. On the first-order functional calculus and the truncation of models. Studia Logica 7 (1958), 181–220. (Polish and Russian summaries)

The author uses notions developed within the theory of models to study a number of aspects of the first order functional calculus. Gödel's theorem is applied to obtain a class of Boolean algebras homomorphically adequate for the functional calculus with added axioms. From a new characterization of theorems of the first order functional calculus, the author obtains the decidability of the monadic calculus. Finally, the completeness of the first order functional calculus with Hilbert's ε-operator is established. The theory of models is extended in the consideration of a number of new concepts, including those of generalized model, and truncation. *E. J. Cogan* (Bronxville, N.Y.)

1265:

Robinson, Abraham. Relative model-completeness and the elimination of quantifiers. Dialectica 12 (1958), 394–407. (German and French summaries)

"Most of the early proofs of decidability or completeness of certain mathematical theories were based on the method of eliminations of quantifiers. Various more recent results on completeness were obtained independently of such procedures. However, it is shown in the present paper that, conversely, the completeness of a mathematical theory will in certain circumstances entail the existence of an elimination method. The proof involves the application of the extended first ε-theorem of Hilbert-Bernays." (Author's abstract) *E. J. Cogan* (Bronxville, N.Y.)

1266:

Porte, J. Deux systèmes simples pour le calcul des propositions. Publ. Sci. Univ. Alger. Sér. A 5 (1958), 5–16.

This is a well written article, for the most part expository, on various formalizations of the classical propositional calculus.

The primitive connectives are implication and negation. Proof is based on the rule of detachment and finitely many axiom schemata. All systems have been introduced previously in the literature; for some of them, however, the present paper probably furnishes the first proof of independence of axioms.

It is shown that two formalizations, one due to Curry [*Leçons de logique algébrique*, Gauthier-Villars, Paris, 1952; MR 13, 613] and the other implicit in theorems of Tarski [*Logic, semantics, metamathematics. Papers from 1923 to 1938*, Clarendon Press, Oxford, 1956; MR 17, 1171; Ch. III] give particularly simple proofs of the principal formal theorems of the two-valued calculus.

G. F. Rose (Santa Monica, Calif.)

1267:

Borkowski, Ludwik. Systems of the propositional and of the functional calculus based on one primitive term. Studia Logica 6 (1957), 7–55. (Polish and Russian summaries)

Four formal systems (SI–SIV) of the functional calculus and one (SV) of the extended propositional calculus (with truth-functor variables and their quantification) are considered. Each one of these systems is based on two or more primitive operators. For each system the author finds an equivalent system based on a single primitive operator.

Systems SI and SII are based on disjunction and universal quantifier as primitives. These are respectively equivalent to systems S1 and S2 based on a "universal

disjunctive quantifier" Π_x^D, for which $\Pi_x^D fxgx$ may be interpreted as "for each x, either not fx or not gx". S1 has rules of inference and no axioms; S2 has rules and one axiom schema.

Systems SIII–SV have implication and universal quantifier as primitives, and SIII has also negation. These are respectively equivalent to systems S3–S5 based on a "universal quantifier with restricted range" Π_x^C, for which $\Pi_x^C fxgx$ may be interpreted as "for each x, if fx then gx". System S3 is based on rules alone; S4 and S5 each have one axiom schema.

G. F. Rose (Santa Monica, Calif.)

1268:

Borkowski, Ludwik; and Słupecki, Jerzy. A logical system based on rules and its application in teaching mathematical logic. Studia Logica 7 (1958), 71–113. (Polish and Russian summaries)

The authors describe four systems in which the notion of formal proof is based solely on rules. These systems are equivalent respectively to the classical propositional calculus, the first-order functional calculus, the Heyting intuitionistic propositional calculus and the Lewis system S4 of strict implication [cf. Lewis and Langford, *Symbolic logic*, Century, New York-London, 1932].

Regarding the ease of constructing proofs and the extent to which intuitive notions are represented by the formalism, the authors feel that these systems are superior to others.

The four systems have two broad classes of rules: rules of introduction or elimination of logical functors; rules for constructing suppositional proofs. The first class includes the rule of detachment, which is common to all four systems. In addition, the classical systems have the rules $F\ G/F\cdot G$, $F\cdot G/F$, $F\cdot G/G$, $F/F\vee G$, $G/F\vee G$, $F\vee G\ \sim F/G$ and the obvious rules for equivalence. The intuitionistic propositional calculus has the additional rule

$$F\to H\ G\to H\ F\vee G/H.$$

Introduction and elimination of universal quantifier and introduction of existential quantifier are the same as Gentzen's *AB*, *AE* and *EE* [Math. Z. **39** (1934/35), 176–210, 405–431]. The rule of existential quantifier elimination consists of deleting $\sum_a$ from $\sum_a F$ and substituting for the variable a a constant s with subscripts $b_1, \cdots, b_n$ where $b_1, \cdots, b_n$ are the free variables of F distinct from a. In the system of strict implication, the rules for conjunction and equivalence are the same as in the classical system, while the rules for $\diamond$ ("possibility") are $\sim(F \prec \sim F)/\diamond F$, $\diamond F/\sim(F\prec\sim F)$.

A rule of suppositional proof may be either direct or apagogical (i.e. contradictory). In the classical systems the direct rule proves a formula of form

(I) $\quad F_1\to(F_2\to\cdots\to(F_n\to F)\cdots)\quad (n\geq 1)$

by means of a sequence whose first n members are the "suppositions" $F_1, \cdots, F_n$, whose other members are either formulas already proved or results of applying introduction or elimination rules to preceding members, and whose last member is F. The apagogical rule proves (I) (with $n\geq 0$) by means of a sequence beginning with $F_1, \cdots, F_n, \sim F$ and ending when two members of form G and $\sim G$ are included, the rule for generating new members being stated as before. The direct rule turns out to be derivable from the other rules.

For the intuitionistic propositional calculus, the direct rule is changed by allowing n to be zero and allowing F to be added if it is identical with a supposition. The apagogical rule proves formulas of the form $F_1\to(F_2\to\cdots\to(F_n\to\sim F)\cdots)$ from suppositions $F_1, \cdots, F_n, F$, the sequence being generated as in the direct case.

The system of strict implication is based on an apagogical rule. Allowing for differences in the rules of introduction and elimination, this rule is stated in the same way as its classical counterpart except that each supposition $F_1, \cdots, F_{n-1}$ must be an impossibility, an implication or an equivalence.

Each system is extensively investigated for derived rules and alternative formulations.

G. F. Rose (Santa Monica, Calif.)

1269:

Borkowski, Ludwik. On modal terms. Studia Logica 7 (1958), 7–41. (Polish. Russian and English summaries)

A propositional metacalculus is proposed as enlarged by quantifying (i) propositional variables (according to Łukasiewicz) and (ii) the s.c. propositional indices (according to the author).

The aim is to reach a formalism of the propositional noncontradictoricity, or—modally speaking—of possibility. The author's informal explanations are not clear, and taken strictly, erroneous. The idea, however, is correct and clear but essentially not new [comp., e.g. Wajsberg, Monatsh. Math. Phys. **40** (1933), 113–126; Carnap, J. Symbolic Logic **11** (1946), 33–64; MR **8**, 429]. In the author's notation, it seems to be as follows. Let Φ be a propositional function and ‚v' (the author's s.c. propositional index) denote an ordinary truth-evaluation. Let Φ_v state that Φ becomes true under v. Then the index-quantified expressions $\sum_v \Phi_v$ [$\prod_v \Phi_v$] mean that Φ is not contradictory [resp. Φ is identical]. Syntactically (and with the same effect as before), one could also take v for a definite simultaneous replacement of propositional variables by propositional functions. (This perhaps was the author's intention.) Another interpretation (also mentioned by the author): ‚v' is the common individual variable, and if ‚p' is a class-variable, then ‚p_v' (usually ‚$p(v)$') is the corresponding atomic expression of the s.c. homogenous class calculus. (After a transportation in Φ of the v's to the propositional variables, Φ_v becomes a molecular expression of the homogeneous class calculus.)

Various forms of calculi of this "v-quantification" are introduced in order to represent modal terms of Lewis' system S5 in a strictly formal sense. A zero-one decision process for the interpreted modal terms is given.

L. Rieger (Prague)

1270:

Suszko, Roman. A formal theory of the logical values. I. Studia Logica 6 (1957), 145–237. (Polish. Russian and English summaries)

An exposition of the simplest well-known facts of the many-valued propositional logic. *L. Rieger* (Prague)

1271:

Evans, Trevor; and Hardy, Lane. Sheffer stroke functions in many-valued logics. Portugal. Math. 16 (1957), 83–93.

The authors describe a new class of Sheffer stroke functions for Post's n-valued logic and give a simple characterisation of another class of these functions. They also discuss the corresponding problem for some Łukasiewicz logics. Some of the results in this connection had already been obtained by the reviewer [Proc. Cambridge Philos. Soc. **48** (1952), 369–373; MR **14**, 3], though the authors were unaware of this until after their paper was written. *A. Rose* (Nottingham)

1272:

★Goodstein, R. L. Recursive number theory: A development of recursive arithmetic in a logic-free equation calculus. North-Holland Publishing Company, Amsterdam, 1957. xii+190 pp. 18 guilders.

This monograph is a detailed and carefully written elaboration of Skolem's remark that much of elementary number theory can be developed within a quantifier-free formal system in which definitions by primitive recursion are permitted. An elegant feature of the present development is that not even propositional connectives need be taken as primitive. After showing how to define various functions within his "equation calculus," the author proceeds to give formal proofs of some of the basic theorems of elementary number theory. Various reductions of the defining schemata for primitive recursive functions are developed. A final chapter gives a proof of Gödel's incompleteness theorem and of Skolem's theorem on the existence of non-standard models in arithmetic. There are numerous exercises. *M. Davis* (Hartford, Conn.)

1273:

Kleene, S. C. Recursive functionals and quantifiers of finite types. I. Trans. Amer. Math. Soc. **91** (1959), 1–52.

Variables for natural numbers are said to be of type 0, and variables for functions whose arguments are of type at most $n-1$ are said to be of type n. One can establish a hierarchy of arithmetic predicates by considering quantification over variables of increasing type. Within each category of this hierarchy, another hierarchy may be established according to the number of alternations of existential and universal quantifiers in the quantifier prefix of the definition of a predicate. Such hierarchies have been investigated for type 0 and type 1 variables. To establish them for higher types, and to investigate an extension of the notion of degree of unsolvability to predicates with higher type quantifications, Professor Kleene develops the theory of general and partial recursiveness for such predicates, leaving the development of the hierarchy for a future part II of the present paper. The extension of the notions of general and partial recursiveness to predicates of higher type entails a number of reformulation of notions related to the cases of types 0 and 1 quantifications. *E. J. Cogan* (Bronxville, N.Y.)

1274:

Péter, Rózsa. Graphschemata und rekursive Funktionen. Dialectica **12** (1958), 373–393.

The author considers the problem of determining the greatest class of functions definable by purely constructive means which includes special recursive functions and is included among general recursive functions. Historically, the search for such a class has faltered at some point. In this paper an attack on the problem through the medium of block diagrams similar to those used in machine programming also falters. The attempt is to arrange appropriately defined graph schemes into equivalence classes which can be used as degrees of complexity ranging over the interval between special recursive and general recursive classes. However, this method does not yield the desired result for the following theorem holds: The class of partial recursive functions coincides with the class of functions defined by normal graph schemes; and the class of general recursive functions of k arguments with the class of all functions of k arguments whose value for each natural number k-tuple can be defined by a normal graph scheme. *E. J. Cogan* (Bronxville, N.Y.)

1275:

Gödel, Kurt. Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. Dialectica **12** (1958), 280–287. (English summary)

It follows from the author's results that Hilbert's finitary standpoint does not make allowance for a suitable framework for metamathematics. Bernays has advocated an extension of this standpoint by the admission of suitable abstract concepts besides combinatorial concepts referring to symbols. The author first discusses in general the notion of an abstract (as opposed to "anschaulich") concept. He makes a distinction between two aspects of the finitary standpoint, which can be characterised as the aspect of effective constructibility and the specifically finitistic aspect of only admitting constructions which are "anschaulich" in a spatio-temporal sense.

It is the second aspect which has to be abandoned. So far this requirement has been met by means of an adjunction of parts of intuitionistic logic and of the theory of ordinals.

Now the author develops another possibility which hitherto has not been applied. It uses the notion of a computable function of finite type t, which is recursively defined, as follows: (1) computable functions of type 0 are the natural numbers; (2) the notions of a computable function of types $t_0, t_1, \cdots, t_k$ $(k>0)$ being defined, we define a computable function of type $(t_0, t_1, \cdots, t_k)$ to be an operation which, with each k-tuple of computable functions of types $t_1, \cdots, t_k$, effectively associates a computable function of type t_0. Taking the type t as a variable, we obtain the abstract notion of a computable function of arbitrary finite type t.

A formal system T is introduced which differs from the system of primitive recursive number theory only by the fact that the variables (with the exception of those to which induction is applied) may have an arbitrary finite type over the natural numbers. The system T is equivalent to a system of recursive number theory in which complete induction over all ordinals $<\varepsilon_0$ is permitted. Now the consistency of classical number theory reduces to the consistency of the system T in two steps: (1) as shown previously by the author (1933), the consistency of classical number theory reduces to that of Heyting's intuitionistic number theory; (2) Heyting's intuitionistic arithmetic admits of an interpretation in the system T (presumably this interpretation differs from the intended one).

To conclude, the author remarks that, on similar lines, much stronger systems can be obtained, for instance, by

the admission of transfinite types or of the methods applied by Brouwer in proving his "fan theorem".
E. W. Beth (Amsterdam)

COMBINATORIAL ANALYSIS
See also 1673, 1693.

1276:
Fleĭšman, B. S. Combinatorial analysis of arrangements. Moskov. Oblast. Pedagog. Inst. Uč. Zap. **57** (1957), 173–198. (Russian.)

This paper considers the possible arrangements of a different objects in some or all of s different places, where only one object can be in each place, but the same object may be in different places. Such an arrangement is considered as a mapping from the set of objects into a subset of the set of places. This model is used, with the help of generating functions, to derive certain elementary relations among multinomial coefficients. (These relations could have been derived by general reasoning from first principles or, as the author points out, by algebraic manipulation of multinomial expansions.) Relations are derived between different types of generating functions for numbers of arrangements, and applications to probability theory, atomic physics, and information theory are mentioned. *T. N. E. Greville* (Kensington, Md.)

ORDER, LATTICES
See also 1282, 1355, 1356, 1357.

1277:
Guillaume, Marcel. Sur les topologies définies à partir d'une relation d'ordre. Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° **29** (1956), no. 6, 42 pp.

Soit $Sg(x)$ resp. $Sd(x)$ l'ensemble des minorants resp. majorants d'un élément x dans un ensemble ordonné E. L'ensemble $F \subset E$ est une partie à gauche resp. à droite si $x \in F$ entraîne $Sg(x) \subset F$ resp. $Sd(x) \subset F$. L'auteur introduit la topologie gauche resp. droite de E comme la topologie Tg resp. Td dont les ensembles ouverts sont les parties à gauche resp. à droite de E. Une partie $F \subset E$ est appelée d'ordre nul à droite [resp. nul à gauche, resp. nul] si les bornes supérieures [resp. inférieures, resp. supérieures et inférieures] existantes dans E des parties totalement ordonnées non vides de F appartiennent à F. La topologie dont les ensembles fermés sont les parties d'ordre nul à droite [resp. nul à gauche, resp. nul] de E est appelée la topologie longitudinale à gauche [resp. longitudinale à droite, resp. longitudinale] et désignée T_- [resp. T_+, resp. T_l]. L'auteur étudie les propriétés de ces topologies et leurs relations avec la topologie d'intervalles et avec la topologie d'ordre. Tg et Td ont pour borne supérieure la topologie discrète. T_- et T_+ ont pour borne inférieure la topologie T_l. T_- resp. T_+ est plus fine que Tg resp. Td. La topologie d'intervalles est moins fine que T_l. Les topologies T_l, T_+, T_- sont séparées. La topologie T_l d'un ensemble ordonné produit est plus fine que la topologie produit des topologies T_l des facteurs.

Une partie $I \subset E$ est dite intègre à droite [resp. gauche] si toute partie totalement ordonnée non vide de I et possédante dans I une borne supérieure [resp. inférieure] possède dans E une borne supérieure [resp. inférieure] égale. I est intègre si elle est à la fois intègre à droite et intègre à gauche. L'auteur prouve le théorème: Tout ensemble ordonné est isomorphe à une partie intègre: (a) du réseau complet $\cup$-distributif général de ses parties à gauche d'ordre nul; (b) du réseau complet $\cap$-distributif général de ses parties à gauche ouvertes pour T_l; (c) d'un réseau complet distributif général. *J. Jakubík* (Košice)

1278:
Kolibiar, Milan. On the axiomatic of modular lattices. Czechoslovak Math. J. **6 (81)** (1956), 381–386. (Russian. English summary)

Ein modularer Verband wird als eine abstrakte Algebra S mit zwei binären Operationen $\cap$, $\cup$ charakterisiert, welche den folgenden Axiomen genügen: (P1) Für beliebige Elemente a, b, c, d von S ist

$$[(a\cap b)\cap c]\cup(a\cap d) = [(d\cap a)\cup(c\cap b)]\cap a.$$

(P2) Für jedes a, b in S ist $[a\cup(b\cap b)]\cap b = b$. Diese Axiome sind unabhängig. Ähnlich wird ein modularer Verband mit einem größten Element charakterisiert; das Axiom (P2) wird dabei durch ein anderes ersetzt. *M. Novotný* (Brno)

1279:
Riečan, J. Zu der Axiomatik der modulären Verbände. Acta Fac. Nat. Univ. Comenian. Math. **2** (1958), 257–262. (Slovak. Russian and German summaries)

Ein modularer Verband wird als eine abstrakte Algebra S mit zwei binären Operationen $\cap$, $\cup$ charakterisiert, welche den folgenden Axiomen genügen: (R1) Für beliebige Elemente a, b, c von S ist

$$(a\cap b)\cup(a\cap c) = [(c\cap a)\cup b]\cap a.$$

(R2) Für beliebige Elemente a, b, c von S ist

$$[a\cup(b\cup c)]\cap c = c.$$

Diese Axiome sind unabhängig. [Vgl. Kolibiar #1278.]
M. Novotný (Brno)

1280:
Pierce, R. S. A note on complete Boolean algebras. Proc. Amer. Math. Soc. **9** (1958), 892–896.

Let B be a Boolean algebra. B is called homogeneous if for every $a \in B$ ($a \neq 0$) the Boolean algebra $\{x;\ x \leq a,\ x \in B\}$ is isomorphic with B. B. Jónsson has asked the question what the possible cardinalities are of complete homogeneous Boolean algebras. In the paper under review the author answers this question in an even more general form. The paper starts with a general decomposition theorem. On the basis of this theorem it is proved that if B is an infinite, complete Boolean algebra of cardinality $\aleph$, then $\aleph^{\aleph_0} = \aleph$. On the other hand it is shown that for every infinite cardinal number $\aleph$, satisfying $\aleph^{\aleph_0} = \aleph$, there exists a complete, homogeneous Boolean algebra B of cardinality $\aleph$. In fact, B is the complete, homogeneous Boolean algebra of regular open subsets of the Cantor space $2^{\aleph}$. Thus we can also say that B is the completion by cuts of the free Boolean algebra of $\aleph$ generators. Under the assumption of the generalised continuum hypothesis,

the cardinal numbers $\aleph_\alpha$ for which $\aleph_\alpha^{\aleph_0}=\aleph_\alpha$ are characterised as follows. $\aleph_\alpha^{\aleph_0}=\aleph_\alpha$ if and only if one of the following conditions holds: (i) α is not a limit ordinal; (ii) α is a limit ordinal which (considered as a well ordered set) contains no countable cofinal subset [A. Tarski, Fund. Math. **7** (1925), 1–14]. *Ph. Dwinger* (Lafayette, Ind.)

GENERAL MATHEMATICAL SYSTEMS

See also 1331.

1281:

Kogalovskiĭ, S. R. Universal classes of algebras. Dokl. Akad. Nauk SSSR **122** (1958), 759–761. (Russian)

A class K of algebras is said to be locally defined provided that an algebra belongs to K if and only if each of its finitely generated subalgebras belongs to K. This usage of terminology is broader than Mal'cev's [same Dokl. **108** (1956), 187–189; MR **18**, 107]. A class of algebras is universal if and only if it is locally defined and closed. Further results give other properties of universal classes of algebras and of subclasses of K which are universal relative to K. *R. A. Good* (College Park, Md.)

1282:

Lowig, H. F. J. On some representations of lattices of law relations. Osaka Math. J. **10** (1958), 159–180.

This paper is concerned with the properties of certain operations and relations in the theory of freely generated algebras which are not named by the author, but are defined in a notation developed in earlier papers [J. Reine Angew. Math. **190** (1952), 65–74; **193** (1954), 129–142; Proc. Cambridge Philos. Soc. **53** (1957), 790–795; MR **14**, 443; **16**, 786; **19**, 1153], on which the present paper leans so heavily that it is not possible to quote any particular results in a review. *H. A. Thurston* (Vancouver, B.C.)

1283a:

Ghika, Al. Structures algébriques ternaires. Com. Acad. R. P. Romîne **8** (1958), 257–261. (Romanian. Russian and French summaries)

1283b:

Ghika, Al. Structures algébrico-topologiques binaires-ternaires. Com. Acad. R. P. Romîne 8 (1958), 447–450. (Romanian. Russian and French summaries)

In the first paper the author calls an n-ary operation $(x_1, x_2, \cdots, x_n)\to x_1x_2\cdots x_n$ in a set n-associative if for each $x_1, \cdots, x_{2n-1}$ we have

$$(x_1\cdots x_n)x_{n+1}\cdots x_{2n-1} = x_1(x_2\cdots x_{n+1})x_{n+2}\cdots x_{2n-1} = \cdots = x_1\cdots x_{n-1}(x_n\cdots x_{2n-1}).$$

He constructs examples to show that this concept has its origin in some problems of non-associative binary systems. The first paper deals with spaces of functions with no other restrictions; in the second paper the functions are supposed to be homomorphisms of other algebraic systems. *M. M. Day* (Urbana, Ill.)

CLASSICAL ALGEBRA

See also 1927.

1284:

★Derwidué, L. Introduction à l'algèbre supérieure et au calcul numérique algébrique. Masson et Cie., Paris; Sciences et Lettres, Liège; 1957. 431 pp. 6000 fr.

This book combines the areas of classical numerical analysis, classical theory of equations, and higher algebra into a very unusual combination. Starting at "The mechanization of algebraic calculation", it first gives a description of how to use a desk calculator in manipulating numbers and then polynomials. It then describes computations with complex numbers, classical manipulations with determinants and those methods of solution of linear equations that prove most useful with desk calculators. Almost no discussion of iterative procedures is given.

A chapter on polynomials includes their description in interpolatory form, the classical solution of equations of the third and fourth degree, and decomposition of rational fractions.

The chapter on the theory of systems of algebraic equations contains a very thorough description of the theory for systems of two equations in two unknowns, and terminates with a brief introduction to algebraic geometry, as applied to more complicated systems. Accompanying it is a very thorough account of the present state of knowledge of techniques for numerical solution of such systems, with a number of problems worked out by hand. The remaining chapters of the book, with one exception, give a formal introductory account of abstract group theory, and an introduction to the theory of finite, symmetric and linear groups, and rings, ideals, fields, vector spaces, and algebras.

The one remaining chapter gives a thorough account of the stability criteria of Routh, Hurwitz, and Schur, and the appendix adds some recent work of the author's.

The book seems aimed at scientists or engineers who are faced with manipulation of polynomials in practical situations, probably servomechanism designers, because of the emphasis on stability. It does not, unfortunately, relate any of the ideas or techniques to large-scale digital computing machinery, or describe effects of inherent or propagated error on solutions. Finally, because the final theory comes at the end with no motivation, it leaves the reader unsatisfied as to how it fits into the whole. Nevertheless, for this class of engineers and those numerical analysts who would like an algebraic source-book of theory on almost any type of work with polynomials, this deserves consideration.

J. W. Carr, III (Chapel Hill, N.C.)

THEORY OF NUMBERS

See also 1272, 1350.

1285:

Errera, A. Une modification de la démonstration de Landau du théorème des nombres premiers. Mathesis **67** (1958), 321–337.

The author gives a simplification of Landau's last proof of the prime number theorem [S.-B. Preuss. Akad. Wiss.

H. **32/33** (1932), 514–521]. It is similar to the author's paper "Sur le théorème fondamental des nombres premiers" [Colloque sur la théorie des nombres, Bruxelles, 1955, pp. 111–118, Thone, Liège, Masson, Paris, 1956; MR **18**, 112], but now the proof is carried through in all details. It is clear and even accessible to a student who may never have heard of the Riemann ζ-function before. The author shows in particular

$$\pi(x) = x/\log x + O(x/\log^2 x) \quad \text{if} \quad x \to \infty.$$

J. Popken (Minneapolis, Minn.)

1286:

Wright, E. M. A definite integral in the asymptotic theory of partitions. Proc. London Math. Soc. (3) **8** (1958), 312–320.

In an earlier paper [same Proc. **7** (1957), 150–160; MR **19**, 16], the author found an asymptotic formula for the number of partitions of a bi-partite number (m, n), with m,n large but of the same order of magnitude. In two of his cases, $I(m/n)$ occurs in the asymptotic formula, where

$$I(z) = \int_0^\infty h(u, z)du, \quad \Re(z) \neq 0,$$

$$h(u, z) = \frac{1}{u(e^{zu}-1)(e^u-1)} - \frac{1}{zu^3} + \frac{z+1}{2zu^2} - \frac{2c_2(z)}{e^u-1},$$

$$c_2(z) = \frac{1}{24}\left(z+3+\frac{1}{z}\right).$$

In the present paper, the author finds (i) an asymptotic formula for $I(z)$ for small z, (ii) simple expressions giving $I(z)$, for all real, positive z, with error $<10^{-5}$, and (iii) an exact formula for $I(z)$ for rational z.

N. J. Fine (Princeton, N.J.)

1287:

Yin, Wen-lin. On Dirichlet's divisor problem. Sci. Record (N.S.) **3** (1959), 6–8.

Let

$$\Delta(x) = \sum_{n\leq x} d(n) - x\log x - (2\gamma-1)x,$$

and let θ be the inf. of the positive numbers α such that $\Delta(x) = O(x^\alpha)$. It was proved by T. T. Chin [Sci. Rep. Tsing Hua Univ. **5** (1950), 402–427] and H.-E. Richert [Math. Z. **58** (1953), 204–218; MR **15**, 11] that $\theta \leq 15/46$. In analogy with a result of L. K. Hua [Quart. J. Math. **13** (1942), 18–29; MR **4**, 190] on the circle problem, it is announced in the present paper that $\theta \geq 13/40$. There are brief remarks on the method of proof, indicating that it is similar to Hua's.

W. J. LeVeque (Ann Arbor, Mich.)

1288:

Erdös, P.; Rényi, A.; and Szüsz, P. On Engel's and Sylvester's series. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **1** (1958), 7–32.

Every number x with $0<x<1$ can be expanded into an Engel's series

$$x = \frac{1}{q_1} + \frac{1}{q_1q_2} + \cdots + \frac{1}{q_1q_2\cdots q_n} + \cdots,$$

where the $q_n = q_n(x)$ are integers such that $2 \leq q_n \leq q_{n+1}$. The authors study the metrical properties of the sequence $q_n(x)$. They consider the interval $0<x<1$ as the space of elementary events and interpret the Lebesgue measure of a measurable subset of (0, 1) as its probability. It is shown that the random variables $x_1 = \log q_1$, $x_n = \log(q_n/q_{n-1})$ are "almost independent" and "almost identically distributed". From this the authors derive for $\log q_n = x_1 + x_2 + \cdots + x_n$ first the central limit theorem (i.e., that the distribution of $n^{-1/2}(\log q_n - n)$ tends for $n \to \infty$ to the normal distribution). Then they prove the strong law of large numbers (i.e., that for almost all x we have $\lim_{n\to\infty} q_n^{1/n} = e$) and, finally, even the law of the iterated logarithm.

The second result has been announced earlier without proof by É. Borel [C. R. Acad. Sci. Paris **225** (1947), 773; MR **9**, 292]; the first and third results are due to P. Lévy [ibid. **225** (1947), 918–919; MR **9**, 292]. However in this paper the authors use a different approach and they give detailed proofs.

They obtain new results by applying their methods to Sylvester's series

$$x = \frac{1}{Q_1} + \frac{1}{Q_2} + \cdots + \frac{1}{Q_n} + \cdots \quad (0 < x < 1),$$

where the Q_n are integers such that $Q_{n+1} \geq Q_n(Q_n-1)+1$. They find that the central limit theorem holds for $\log(Q_n/Q_1Q_2\cdots Q_{n-1})$ and further that $\lim_{n\to\infty} 2^{-n}\log Q_n$ exists and is finite and positive for almost all x in (0, 1).

The paper ends with a discussion of some number-theoretic questions concerning these series; several unsolved problems are mentioned.

J. Popken (Berkeley, Calif.)

1289:

Békéssy, A. Bemerkungen zur Engelschen Darstellung reeller Zahlen. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **1** (1958), 143–151.

The author extends some results obtained in the paper reviewed above. Let x be any number in the interval $0<x<1$ and let

$$x = \frac{1}{q_1} + \frac{1}{q_1q_2} + \cdots + \frac{1}{q_1q_2\cdots q_n} + \cdots$$

denote its expansion into an Engel's series. Let $p_n(k)$ denote the probability $\mathrm{P}\{q_n(x)=k\}$ and let $p_n(k|j)$ stand for $\mathrm{P}\{q_{n+\nu}=k|q_\nu=j\}$ where $\mathrm{P}\{A|B\}$ denotes the conditional probability of A with respect to the condition B. It follows easily $p_n(k) = p_n(k|2)$. The author derives several asymptotic fomulae for $p_n(k|j)$ and for $W_n(k|j) = \sum_{l=k}^\infty p_n(l|j)$ too complicated to reproduce here. From one of his results it follows, e.g., that

$$p_n(k) \sim \Gamma\left(2 - \frac{n-1}{\log k}\right)\frac{(\log k)^{n-1}}{k^2(n-1)!}$$

for $(n-1)/2\log k \leq \theta < 1$. Another result (formula (11C)) gives an upper bound for the absolute value of the remainder in the central limit theorem for $\log q_n$ (see the preceding review).

J. Popken (Berkeley, Calif.)

1290:

Erdös, P.; Szüsz, P.; and Turán, P. Remarks on the theory of diophantine approximation. Colloq. Math. **6** (1958), 119–126.

For fixed $A>0$, $c>1$ and integer $N\geq 2$, denote by $S(N, A, c)$ the set of $\alpha \in [0, 1]$ such that for some integers

x and y with $(x, y)=1$ and $N \leq y < cN$, the inequality $|\alpha - x/y| < A/y^2$ holds. The question is considered whether $\lim_{N\to\infty}$ meas $S(N, A, c) = f(A, c)$ exists, and if so, what its nature is. It is shown that for $0 < A < c/(1+c^2)$, $f(A, c) = 12A\pi^{-2} \log c$, and that for $A > 10$ and $c > 10$ (for example), meas $S(N, A, c)$ is bounded away from 0 and 1 as $N \to \infty$, explicit bounds being given. From the lower bound it follows immediately that if $R(N, c)$ is the set of $\alpha \in [0, 1]$ for which the interval $N \leq y \leq cN$ contains the denominator q_i of at least one convergent to the regular continued fraction expansion of α, then $\lim \inf_{N\to\infty}$ meas $R(N, c) \geq 3\pi^{-2} \cdot (1-c^{-2})$. *W. J. LeVeque* (Ann Arbor, Mich.)

1291:

Poitou, G. Sur les fractions continues arithmétiques. Bull. Soc. Math. Belg. **9** (1957), 3–7.

A discussion of continued fraction expansions (in the rational and imaginary quadratic number fields) whose convergents are also best approximations.

W. J. LeVeque (Ann Arbor, Mich.)

1292:

Sós, Vera T. On a geometrical theory of continued fractions. Mat. Lapok **8** (1957), 248–263. (Hungarian. Russian and English summaries)

Further application is made of a geometric interpretation of continued fractions introduced in an earlier paper [Acta Math. Acad. Sci. Hungar. **8** (1957), 461–472; MR **20** #34]. The method is used to prove the following theorem. Let α have the continued fraction expansion

$$\frac{1}{a_1+}\frac{1}{a_2+}\cdots,$$

with convergents p_k/q_k. For positive integer N determine integers k and r by the inequalities $q_{k-1}+rq_k \leq N < q_{k-1}+(r+1)q_k$, $0 < r \leq a_k$. Then

$$\min_{q\leq N}\left|\alpha-\frac{p}{q}\right| = \begin{cases} \left|\alpha-\dfrac{p_k}{q_k}\right| & \text{if } 0 < r < \Sigma, \\ \left|\alpha-\dfrac{p_{k-1}+rp_k}{q_{k-1}+rq_k}\right| & \text{otherwise,}\end{cases}$$

where

$$\Sigma = \frac{a_k}{2}+\left|\frac{q_{k+1}\alpha-p_{k+1}}{q_k\alpha-p_k}\right|-\frac{q_{k-1}}{q_k}.$$

W. J. LeVeque (Ann Arbor, Mich.)

1293:

Makai, E. An estimation in the theory of diophantine approximations. Acta Math. Acad. Sci. Hungar. **9** (1958), 299–307.

Let $z_1, \cdots, z_n$ be n complex numbers with $1 = z_1 \geq |z_2| \geq \cdots \geq |z_n|$, and put $s_\nu = b_1 z_1^\nu + \cdots + b_n z_n^\nu$. P. Turán [*On a new method in analysis*, Budapest, Akad. Kiadó, 1953; MR **15**, 688] has shown that there is an absolute constant $A > 0$ such that

$$\operatorname*{Max}_{m+1\leq\nu\leq m+n} |s_\nu| \geq \left(\frac{n}{A(m+n)}\right)^n \min_{1\geq j\geq n} |b_1+\cdots+b_j|$$

for all integers $m \geq 0$, $n > 0$. Turán and Vera T. Sós [same Acta **6** (1955), 241–255; MR **17**, 1061] showed that

$$24 \approx 2\exp(1+4/e) \geq A \geq 1.321,$$

the second inequality holding even under the restriction $b_1 = \cdots = b_n = 1$. By further development of another idea of that paper, the present author shows that $A \geq 1.473$, also in the restricted case. The proof depends on an asymptotic estimate for the least of the moduli of the zeros of the nth section of the power series expansion of $\exp(-z^2+2\lambda z)$, where λ is a zero of the Hermite polynomial $H_{n+1}(z)$. *W. J. LeVeque* (Ann Arbor, Mich.)

1294:

Uchiyama, S. A note on the second main theorem of P. Turán. Acta Math. Acad. Sci. Hungar. **9** (1958), 379–380.

In the notation of the preceding review, it is shown that $A \geq e$. In the proof, which is simple and constructive, not all the b_j are unity. It is also asserted that the upper bound $2\exp(1+4/e)$ for A can be replaced by $8e \approx 21.75$, using the argument of Turán and Sós.

W. J. LeVeque (Ann Arbor, Mich.)

1295:

Šidlovskiĭ, A. B. A criterion for algebraic independence of the values of a class of entire functions. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 35–66. (Russian)

In several earlier notes the author had announced without proof generalisations of Siegel's results on the transcendency of E-functions satisfying linear differential equations [C. Siegel, *Transcendental numbers*, Princeton Univ. Press, Princeton, N.J., 1949; MR **11**, 330]. The present paper contains the detailed proofs for the following two main results of the author. (I) Let $f_1(z), \cdots, f_m(z)$ be m E-functions such that

$$y_k' = Q_{k0}(z) + \sum_{i=1}^{m} Q_{ki}(z)y_i \quad (k = 1, 2, \cdots, m),$$

where the Q's are rational functions, and let $\alpha \neq 0$ be an algebraic number distinct from the poles of these rational functions. Then the numbers $f_1(\alpha), \cdots, f_m(\alpha)$ are algebraically independent over the field of rational numbers if and only if the functions $f_1(z), \cdots, f_m(z)$ are algebraically independent over the field of rational functions of z. (II) Let $f_1(z), \cdots, f_m(z)$ be m E-functions satisfying the homogeneous differential equations

$$\text{(D)} \qquad y_k' = \sum_{i=1}^{m} Q_{ki}(z)y_i \quad (k = 1, 2, \cdots, m),$$

where the Q's are rational functions, and let $\alpha \neq 0$ be an algebraic number distinct from the poles of these rational functions. Then the numbers $f_1(\alpha), \cdots, f_m(\alpha)$ do not satisfy any homogeneous algebraic equation with rational integral coefficients if and only if the functions $f_1(z), \cdots, f_m(z)$ do not satisfy any homogeneous algebraic equations with coefficients that are polynomials in z.

The proofs of these important results are based on Siegel's method. The essentially new part consists in the following fundamental lemma of the author. (III) Let $f_1(z), \cdots, f_m(z)$ be m fixed integral functions that satisfy a system of differential equations (D) where the Q's are rational functions. For any positive integer n denote by $P_{11}(z), \cdots, P_{1m}(z)$ m polynomials, of degree at most $2n-1$, not all identically zero, such that in the power series for

$$R_1(z) = \sum_{i=1}^{m} P_{1i}(z)f_i(z) = \sum_{\nu=0}^{\infty} a_\nu \frac{z^\nu}{\nu!}$$

all coefficients a_ν with $0 \leq \nu \leq 2mn-n-1$ vanish. Further, denote by $T(z)$ the least common denominator of the Q's, and define further linear forms

$$R_k(z) = \sum_{l=1}^{m} P_{kl}(z) f_l(z) \quad (k = 2, 3, \cdots)$$

in $f_1(z), \cdots, f_m(z)$ with polynomial coefficients $P_{kl}(z)$ by

$$R_k(z) = T(z) \frac{d}{dz} R_{k-1}(z)$$

where $f_1'(z), \cdots, f_m'(z)$ are to be replaced by their linear expressions in $f_1(z), \cdots, f_m(z)$ from (D). There exist positive integers n_0, p, and q independent of n such that, for $n \geq n_0$, the determinant

$$\Delta(z) = |P_{kl}(z)|_{k,l=1,2,\cdots,m}$$

does not vanish identically and is of the form $\Delta(z) = z^{(2m-1)n-m-p-1}\Delta_1(z)$ where $\Delta_1(z) \not\equiv 0$ is a polynomial in z of degree not exceeding $n+p+\frac{1}{2}qm(m-1)-1$.

K. Mahler (Notre Dame, Ind.)

1296:

Kasch, Friedrich. Über eine metrische Eigenschaft der S-Zahlen. Math. Z. **70** (1958), 263–270.

Let ξ denote a given (real or complex) transcendental number; put

$$\omega_n(H, \xi) = \min P(\xi),$$
$$\bar{\omega}_n(\xi) = \operatorname*{Lim\,sup}_{H\to\infty} \log (\omega_n(H, \xi))^{-1}(\log H)^{-1},$$

where the minimum is extended over all polynomials $P(x)$ of degree $\leq n$ with integer coefficients and of height $\leq H$ (≥ 1). Put $\vartheta_n(\xi) = \bar{\omega}_n(\xi)n^{-1}$. S-numbers are those numbers for which $\vartheta(\xi) = \sup \vartheta_n(\xi)$ ($n\to\infty$) is finite. Mahler conjectured: $\vartheta_n(\xi) = 1$ for almost all real ξ and $\vartheta_n(\xi) = \frac{1}{2}$ for almost all complex ξ ($n \geq 1$). Kubilyus [Dokl. Akad. Nauk SSSR **67**(1949), 703–706; MR **11**, 83] proved the conjecture for $n=2$ by means of Vinogradoff's methods. The author gives an elementary treatment of this case $n=2$, which also allows one to deduce the complex analogue, and discusses the possibility of a generalization for $n>2$. [Cf. also the following review.]

J. F. Koksma (Amsterdam)

1297:

Kasch, Friedrich; und Volkmann, Bodo. Zur Mahlerschen Vermutung über S-Zahlen. Math. Ann. **136** (1958), 442–453.

For the notation cf. the preceding review. The authors prove (a) for almost all real ξ we have $1 \leq \vartheta_n(\xi) \leq 2-2/n$ ($n \geq 3$); (b) for almost all complex ξ we have $1/2-1/2n \leq \vartheta_n(\xi) \leq 3/2-2/n$ ($n \geq 3$); these results contain the best estimates attained till now, viz. by LeVeque [Proc. Amer. Math. Soc. **4** (1953), 189–190; MR **14**, 956]. Using Hausdorff's notion of dimension, they refine their results still further. {It seems that the authors don't know the work of D. J. Lock [Thesis, Free Univ. of Amsterdam, 1947; MR **9**, 79].}

J. F. Koksma (Amsterdam)

1298:

Artyuhov, M. L. A method for counting integral points in n-dimensional polyhedrons. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 215–218. (Russian)

1299:

Woods, A. C. The critical determinant of a spherical cylinder. J. London Math. Soc. **33** (1958), 357–368.

Let $F(x_1, x_2, \cdots, x_n)$ be a distance function in R_n. Then

$$\underline{F}(x_1, x_2, \cdots, x_n, x_{n+1}) = \operatorname{Max}(F(x_1, x_2, \cdots, x_n), |x_{n+1}|)$$

defines a distance function in R_{n+1}. Assume the starbody $K: F \leq 1$ is of finite type and has critical determinant $\Delta(K)$. Then the corresponding starbody $\underline{K}: \underline{F} \leq 1$ in R_{n+1} (the cylinder over the base K) is again of finite type and $\Delta(\underline{K}) \leq \Delta(K)$. Mahler [Quart. J. Math. Oxford Ser. **17** (1946), 16–18; MR **7**, 368] proved that if K is the 2-dimensional sphere $x_1^2+x_2^2 \leq 1$, then $\Delta(\underline{K}) = \Delta(K)$; and Chalk, Rogers and Yeh extended that result to all convex bodies K in R_2. Varnavides showed that $\Delta(\underline{K}) = \Delta(K)$ also remains true for the star domain $|x_1x_2| \leq 1$. Rogers and Davenport, however, constructed starbodies K in R_2 for which $\Delta(\underline{K}) < \Delta(K)$. Now the authors prove that, if $K: x_1^2+x_2^2+x_3^2 \leq 1$ is the unit sphere in R_3, then $\Delta(\underline{K}) = \Delta(K)$. Corollary: Call a system of non-overlapping spheres a regular packing if the centres form a lattice, and call it a semiregular packing if the centres form the union of a lattice and a translation of the lattice, which does not form a new lattice. It follows from the theorem that there is no semiregular packing of spheres with a density closer than that of the closest regular packing.

J. F. Koksma (Amsterdam)

FIELDS

1300:

Krull, Wolfgang. Zur Idealtheorie der unendlichen algebraischen Zahlkörper. J. Sci. Hiroshima Univ. Ser. A **21** (1957/58), 79–88.

Soit K un corps de degré infini de nombres algébriques. On organise l'ensemble Σ des valuations ultramétriques (considérées à l'équivalence près) σ de K par la topologie, où les ensembles $\Sigma(\sigma_k)$ des valuations de K prolongeant quelque valuation fixée σ_k de quelque $k \subset K$ de degré fini constituent une base de la famille des ensembles ouverts. Une $\sigma \in \Sigma$ est isolée pour cette topologie si, et seulement s'il existe un $k \subset K$ de degré fini tel que la valuation induite par σ dans k ne se décompose pas dans K/k. Elle est discrète s'il existe un $k \subset K$ de degré fini tel que K/k soit non-ramifiée par rapport à σ.

Soit $\mathfrak{a}$ un idéal de K. Alors, si $\sigma \in \Sigma$, l'idéal $\mathfrak{q}_\sigma(\mathfrak{a})$, engendré par $\mathfrak{a}$ dans le corps K valué par σ, est dit la composante isolée de $\mathfrak{a}$ correspondante à σ, et cette composante est dite essentielle si elle est $\neq (1)$. Soit $\mathfrak{q}_{\sigma,k}(\mathfrak{a})$ l'idéal engendré par $\mathfrak{a}$ dans un corps $k \subset K$ valué par σ. L'auteur prouve, d'abord, que, pour qu'il existe, pour un σ fixé et pour tout idéal entier $\mathfrak{a}$ de K, un corps $k = k_{\sigma,\mathfrak{a}} \subset K$ de degré fini et tel que, pour tout $k' \supseteq k$ ($k' \subset K$) de degré fini, on ait $\mathfrak{q}_{\sigma,k'}(\mathfrak{a}) = \mathfrak{q}_\sigma(\mathfrak{a}) \cap k'$, il faut et il suffit que σ soit isolée et discrète, tandis que si σ est non-isolée et dense, il existe, à la fois, des idéaux $\mathfrak{a}$ satisfaisant à cette condition, et d'autres pour lesquels quel que soit $k' \subset K$ de degré fini, on a $\mathfrak{q}_{\sigma,k'}(\mathfrak{a}) \neq \mathfrak{q}_\sigma(\mathfrak{a}) \cap k'$.

Un idéal entier $\mathfrak{a}$ est dit un idéal de Nakano (ou nakanien) s'il existe un corps $k \subset K$ de degré fini tel que, pour tout $\sigma \in \Sigma$ et pour tout $k' \supset k$, on ait $\mathfrak{q}_{\sigma,k'}(\mathfrak{a}) = \mathfrak{q}_\sigma(\mathfrak{a}) \cap k'$, et K est dit un corps nakanien si tous ses

idéaux entiers le sont. L'auteur montre que K est nakanien si, et seulement si tous les $\sigma \in \Sigma$ sont isolées et discrètes (c'est-à-dire que, pour chaque valuation, il existe un $k \subset K$ de degré fini, à partir duquel elle ne fait qu'élargir son corps résiduel, sans se décomposer ni se ramifier). Si toutes les $\sigma \in \Sigma$ sont discrètes et si, pour tout $\sigma \in \Sigma$ il existe un voisinage V_σ tel que toutes les $\tau \in V_\sigma$ aient un même ordre de ramification (absolu), les seuls idéaux nakaniens sont les idéaux finiment engendrés. Ce résultat est déjà faux si l'on affaiblit légèrement la seconde condition, en demandant seulement que les ordres de ramification des $\tau \in V_\sigma$ aient un multiple commun fini.

M. Krasner (Paris)

1301:

Šafarevič, I. R. The imbedding problem for splitting extensions. Dokl. Akad. Nauk SSSR **120** (1958), 1217–1219. (Russian)

Soit k/Ω une extension galoisienne de degré fini, dont le groupe de Galois soit F, et soit φ un épimorphisme d'un groupe (d'ordre fini) G sur F. Le problème d'immersion consiste à trouver le critère pour qu'il existe des surextensions K/Ω de k/Ω telles que le groupe de Galois de K/Ω soit identifiable avec G de manière que φ s'identifie avec l'épimorphisme naturel des groupes de Galois, auquel cas K/Ω est dite une solution du problème d'immersion de k/Ω pour l'extension $\varphi: G \to F$. N étant le noyau de φ, G est dite une extension décomposable de F s'il existe un sousgroupe F^* de G isomorphe à F et tel que $G = F^*N$.

L'auteur énonce le résultat suivant, qui généralise les résultats analogues de ses travaux antérieurs sur le même problème et sur la construction des extensions avec les groupes de Galois nilpotents ou résolubles donnés:

Pour toute extension galoisienne k/Ω de groupe F et pour toute extension $\varphi: G \to F$ dont le noyau N est nilpotent, le problème d'immersion possède des solutions.

D'après les indications données par l'auteur, sa méthode de démonstration (qu'il esquisse sommairement dans la Note) ne diffère de celle de ses travaux antérieurs que par une relativisation de la notion de corps scholzien et par les légères modifications de ses autres raisonnements que cette relativisation entraîne.

M. Krasner (Paris)

ALGEBRAIC GEOMETRY

1302:

Palman, Dominik. Die Scheitel einer zirkulären Kurve 3. Ordnung. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. **13** (1958), 97–105. (Serbo-Croatian summary)

Theorem: Any circular cubic of genus $p=1$ has sixteen vertices (i.e., sixteen points in which the osculating circle has a four points contact). If $p=0$, the circular cubic has four vertices if it has a node or an isolated double point and one if it has a cusp. The conditions of reality of the vertices are discussed in detail.

L. A. Santaló (Buenos Aires)

1303a:

Mersch, Jacques. Sur la surface du quatrième ordre contenant une quintique rationnelle tracée sur une quadrique. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 839–850.

1303b:

Mersch, Jacques. Sur une transformation birationnelle de l'espace. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 945–963.

The first paper is a detailed study of six birational involutions of a quartic surface into itself. The transformations are defined in terms of a rational quintic curve contained in the quartic surface and lying on a quadric surface.

The second paper is a similar study of a birational transformation defined by a pencil of quartic surfaces containing the quintic curve of the previous paper.

G. B. Huff (Athens, Ga.)

1304:

Porcu, Livio. Il metodo di "piccola variazione" in problemi concernenti le curve algebriche piane reali. II. Period. Mat. (4) **36** (1958), 239–250.

L'a. continua la sua esposizione didattica [Period. Mat. (4) **36** (1958), 156–174; MR **21** #46] del metodo di "piccola variazione" dando alcune delle più elementari applicazioni del metodo allo studio della forma delle cubiche e quartiche piane e a curve di ordine superiore, di cui nel campo reale espone le principali questioni esistenziali, partendo dal teorema di Harnack e servendosi del metodo della conica ausiliaria di Hilbert.

M. Piazzolla-Beloch (Ferrara)

1305:

Marchionna, Ermanno. Varietà di prima specie e varietà intersezioni complete. Boll. Un. Mat. Ital. (3) **13** (1958), 406–417. (English summary)

V_d is an algebraic variety in S_r, possibly reducible, but of pure dimension d and without multiple components. By the virtual arithmetic genus of V_d is meant the coefficient p_d in the characteristic function

$$\chi(V_d, l) = \binom{l+d}{d} + \sum_{i=0}^{d} (-1)^i \binom{l+d-i-1}{d-i} p_i$$

which for sufficiently high values of l gives the postulation of V_d for hypersurfaces of order l. V_d is said to be of the first kind if, V_{d-1} being its section by a generic hyperplane S_{r-1}, every hypersurface of order l in S_{r-1} containing V_{d-1} is the section by S_{r-1} of a hypersurface of order l in S_r containing V_d (for all values of l).

V_d^* being a second variety in S_r, likewise of pure dimension d and without multiple components, the equation ideals of V_d, V_d^* are called equivalent if they contain the same number of linearly independent forms of degree l, for all l; which means that for all l, V_d, V_d^* have the same effective postulation for hypersurfaces of order l.

The main results are the following.

Theorem I: If (A) the equation ideals of the generic hyperplane sections V_{d-1}, V_{d-1}^* are equivalent, (B) the virtual arithmetic genera of V_d, V_d^* are equal, and (C) V_d is of the first species, then V_d^* is also of the first species, and the equation ideals of V_d, V_d^* are equivalent.

Theorem II: Necessary and sufficient conditions for V_d to be the complete intersection of $r-d$ hypersurfaces, of

orders $n_1, \cdots, n_{r-d}$, are that: (a) its virtual arithmetic genus is equal to the geometric genus of an irreducible and non-singular W_d, complete intersection of $r-d$ hypersurfaces of these orders (this genus is expressible as a function, which is given, of r, d, $n_1, \cdots, n_{r-d}$); (b) there exists at least one hyperplane S_{r-1} cutting V_d in a V_{d-1} which is the complete intersection of $r-d$ hypersurfaces of these orders in S_{r-1}. If $d \geqq 2$ and V_d is irreducible, condition (b) alone is sufficient. *P. Du Val* (London)

1306:

Žižčenko, A. B. On the number of subfields of a field of algebraic functions of one variable. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 541–548. (Russian)

In der vorliegenden Arbeit wird folgender Satz bewiesen: Ein Körper Σ algebraischer Funktionen einer Veränderlichen vom Geschlecht $g>1$ über algebraisch abgeschlossenem Konstantenkörper besitzt nur eine endliche Anzahl von Unterkörpern Σ', über denen er separabel ist. Im klassischen Fall bei Charakterik 0 des Grundkörpers ist der Satz von Severi gezeigt worden, was auch dem Verf. zum Muster dient. Der Beweis vollzieht sich in 2 Etappen. Erst wird mit rein geometrischen Methoden unter Benutzung des dreifach kanonischen Modells der Ausgangskurve und bestimmten Projektionen desselben gezeigt, daß die genannte Menge von Unterkörpern algebraisch ist. Daraufhin wird bewiesen, daß diese Menge nulldimensional ist. Es folgen Beispiele, die zeigen, daß der Satz bei Fortfall einer der Voraussetzungen wie $g>1$ oder Separabilität nicht mehr richtig ist.

W. Burau (Hamburg)

1307:

Gröbner, Wolfgang. Über die Parameterdarstellungen algebraischer Mannigfaltigkeiten mittels Liescher Reihen. Math. Nachr. **18** (1958), 360–375.

Die vorliegende Arbeit bringt eine vertiefte und verbesserte Darstellung der schon an anderer Stelle veröffentlichten Ergebnisse [s. Gröbner, Monatsh. Math. **61** (1957), 209–224; MR **20** #877]. Wieder werden der algebraischen Mannigfaltigkeit M_d von d Dimensionen des komplexen P_n gewisse Liesche Operatoren $D_1, \cdots, D_d$ zugeordnet, aus ihnen $D=t_1D_1+\cdots+t_dD_d$ gebildet und damit wiederum die sog. Lieschen Reihen

$$X_k = \sum_{v=0}^{\infty} (D^v/v!)x_k \quad (k = 0, 1, \cdots, n).$$

Diese Reihen konvergieren und liefern für fast alle Punkte (x_i) von M_d lokale Parameterdarstellungen. Dabei brauchen die D_i nicht vertauschbar zu sein, entgegen der früheren Darstellung. Man kann es aber so einrichten, daß sie vertauschbar sind und zu einer abelschen Gruppe mithin gehören. Weiterhin wird gezeigt, daß die Parameter $t_1, \cdots, t_d$ dieser globalen Darstellung sich als abelsche Integrale auf M_d erweisen. Die umgekehrte Aufgabe, zu gegebenen d abelschen Integralen, d. h. Integralen über Differentialformen auf M_d, die Operatoren zu berechnen, führt zur Lösung des Jacobischen Umkehrproblems für die d Integrale. Dies wird am Beispiel $d=1, n=2$ erläutert, d. h. an der bekannten Jacobischen Aufgabe, ein einzelnes abelsches Integral auf einer ebenen algebraischen Kurve umzukehren. Man gelangt zu einer Übersicht über alle Lösungen dieses Problems. *W. Burau* (Hamburg)

1308:

van der Waerden, B. L. Zur algebraischen Geometrie. 19. Grundpolynom und zugeordnete Form. Math. Ann. **136** (1958), 139–155.

[For parts 17 and 18, see same Ann. **128** (1954), 128–134, 135–137; MR **16**, 165.] Let $(x)=(1, x_1, \cdots, x_n)$ be a generic point over k of a variety of dimension $r<n$ in P_n, and u_{ij} n^2 indeterminates. Consider the expressions

$$z_i = \sum_{j=j}^{n} u_{ij}x_j \quad (i = 0, 1, \cdots, n-1).$$

Then there exists (up to factors independent of the Z's) a unique irreducible polynomial with coefficients in k, $F(Z_0\cdots Z_r,\ u_{01}\cdots u_{rn})$, of minimal degree in Z_0 such that $F(z_0\cdots z_r,\ u_{01}\cdots u_{rn})=0$. This polynomial is the "associated form of V". In a very clear and crisp style the author assembles, recasts, and gives new proofs for many results concerning this form mainly due to himself [part 3 of this series, ibid. **108** (1933), 694–698], van der Waerden and Chow [part 11, ibid. **113** (1937), 692–704], Keshava Hegde [Comment. Math. Helv. **30** (1956), 124–138; MR **17**, 665] and Krull [Arch. Math. **1** (1948), 129–137; MR **11**, 310]. He proves, for instance, that the variety V is absolutely irreducible if and only if F is the qth power of an absolutely irreducible form with $q=1$ or $q=p^e$ (p being the characteristic of k), shows how the minimal field of definition of V can be determined from the coefficients of F, and how one obtains from the consideration of F all points of V. *A. Gutwirth* (Evanston, Ill.)

1309:

Andreotti, Aldo. On a theorem of Torelli. Amer. J. Math. **80** (1958), 801–828.

Let Γ [resp. Γ'] be a complete non-singular curve of the same genus g, J [J'] be the Jacobian variety of Γ [Γ'] and Θ [Θ'] be a canonical divisor on J [J'] with respect to Γ [Γ']. The theorem of Torelli discussed here is that Γ and Γ' are birationally equivalent if and only if there is an isomorphism α between J and J' such that $\alpha(\Theta)\equiv\Theta'$ (numerical equivalence). The theorem is due to Torelli in the classical case. The theorem in the abstract case has been treated by Weil and also recently by the reviewer. Here, the author gives another and slightly more general proof of the theorem of Torelli in the abstract case. The main theorem in this paper is that "two curves Γ, Γ' are birationally equivalent if and only if their $(g-1)$-fold symmetric products $(\Gamma)_{g-1}$, $(\Gamma')_{g-1}$ are birationally equivalent". From this the theorem of Torelli follows at once.

The main idea is as follows. Let q be an integer such that $1\leqq q\leqq g-1$; assuming that Γ, Γ' are non-hyperelliptic and that they are projective canonical models, let $(x_1, \cdots, x_q)$ be q-points of Γ in general position. Let P_{q-1} be the linear space spanned by these q-points. Then we get a rational mapping f_q of $(\Gamma)_q$ into the Grassmann manifold, by sending the corresponding point on $(\Gamma)_q$ to $(x_1, \cdots, x_q)$ to the Plücker coordinates of P_{q-1}. If $q=g-1$, the image of $(\Gamma)_{g-1}$ by $f_{g-1}=f$ is the dual projective space P^*_{g-1} to the ambient space P_{g-1} of Γ. The branch locus D of f on P^*_{g-1} (the set of points y on P^*_{g-1}, together with their specializations, of dimension $g-2$ such that $f^{-1}(y)$ is not defined or is defined but contains a component of multiplicity >1) is shown to be irreducible and a generic point

of D is a generic tangent hyperplane to Γ. Showing that if $(\Gamma)_{g-1}$ and $(\Gamma')_{g-1}$ are birationally equivalent, branch loci D, D' of the maps $f:(\Gamma)_{g-1}\to P^*_{g-1}$, and $f':(\Gamma')_{g-1}\to P^*_{g-1}$ coincide, the author concludes that the plane projections of Γ, Γ' from a general linear space of dimension $g-4$ coincide. This proves that Γ and Γ' are birationally equivalent. In the case of characteristic $p>0$, $\neq 2$, the above idea works essentially, but when $p=2$, or when Γ, Γ' are hyperelliptic, the author takes care differently.

T. Matsusaka (Evanston, Ill.)

1310:

Hironaka, Heisuke. A note on algebraic geometry over ground rings. The invariance of Hilbert characteristic functions under the specialization process. Illinois J. Math. **2** (1958), 355–366.

Let V be a projective variety defined over a field k with a discrete valuation v, and let $\bar{V}$ be a reduction of V with respect to v. Then we have a model, which is denoted by $[V]_v$, over the ground ring v, so that it is the set of specialization rings of specializations, in V or $\bar{V}$ over k or v, of a fixed generic point of V over k.

Part I of this paper is concerned with the ideal in a spot of the model $[V]_v$ generated by the prime element of v and proves more than the following: (i) If $\bar{V}$ is $\bar{v}$-normal, then every spot of $[V]_v$ is normal (hence V is k-normal); (ii) if $\bar{V}$ is $\bar{v}$-nonsingular, then every spot of $[V]_v$ is v-unramified regular (hence V is k-nonsingular).

In Part II, the author discusses the condition which should be imposed on the spots of $[V]_v$ in order that the varieties V, $\bar{V}$ (or a k-rational V-divisor Z and its reduction $\bar{Z}$) have the same Hilbert characteristic function. The results are the following.

(1) V and $\bar{V}$ have the same Hilbert characteristic function if and only if the prime element of v generates an unmixed ideal in every spot of $[V]_v$; if $\bar{V}$ is $\bar{v}$-normal, then V and $\bar{V}$ have the same Hilbert characteristic function. [Cf. Igusa, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 34–37, 317–320; MR **17**, 87.]

(2) Z and $\bar{Z}$ have the same characteristic function if and only if the prime element of v and the ideal of Z generate an unmixed ideal in every spot of $[V]_v$; if $\bar{Z}$ does not contain any singular point of $\bar{V}$ in the absolute sense, then they have the same characteristic function, and Z does not contain any singular point of V. [Cf. Matsusaka, J. Math. Soc. Japan **5** (1953), 113–136; MR **15**, 465.]

The following lemma, proved in Part I, is interesting in itself, and is one of the key lemmas of this paper.

Let $\mathfrak{O}$ be a spot (which is an integral domain) and let a be a non-unit of $\mathfrak{O}$. Suppose that (1) $a\mathfrak{O}$ has only one minimal prime ideal $\mathfrak{m}$ and $a\mathfrak{O}_\mathfrak{m}$ is the maximal ideal of $\mathfrak{O}_\mathfrak{m}$; (2) $\mathfrak{O}/\mathfrak{m}$ is a normal ring. Then $\mathfrak{m}=a\mathfrak{O}$, and $\mathfrak{O}$ is a normal ring.

{Reviewer's note: The assumption that $\mathfrak{O}$ is a spot is important only for the validity of the chain problem of prime ideals, hence the lemma can be generalized, but not to arbitrary Noetherian local integral domains.}

M. Nagata (Kyoto)

1311:

Lang, S.; and Néron, A. Rational points of abelian varieties over function fields. Amer. J. Math. **81** (1959), 95–118.

A simplified proof of the theorem of the base for divisors is given in this article. The base theorem asserts, as is well known, that the group of rational divisors over a ground field modulo algebraic equivalence is finitely generated, and this was proved by A. Néron [Bull. Sci. Math. France **80** (1952), 101–166; MR **15**, 151].

The authors reduce the assertion first to the following.

Let K be a finitely generated regular extension of a field k. Let A be an abelian variety defined over K, and let (B, τ) be its K/k-trace, namely, B is an abelian variety, τ is a homomorphism $B\to A$ defined over K, and for any abelian variety C defined over an extension E of k which is free from K over k and homomorphism $\alpha: C\to A$ defined over KE, there exists a homomorphism $\alpha': C\to B$ defined over E such that the diagram

$$\begin{tikzcd} C \arrow[rr, "\alpha"] \arrow[dr, "\alpha'"'] & & A \\ & B \arrow[ur, "\tau"'] & \end{tikzcd}$$

is commutative. Then A_K/B_k is finitely generated. [The notion of K/k-traces was studied by W. L. Chow, Trans. Amer. Math. Soc. **78** (1955), 582–586; MR **17**, 193.]

Then the authors prove the assertion reducing to the case where k is algebraically closed and K is of transcendence degree 1 over k.

Finite generation of the group of rational points of an abelian variety over an algebraic number field is also proved.

M. Nagata (Kyoto)

1312:

Sampson, J. H.; and Washnitzer, G. A Vietoris mapping theorem for algebraic projective fibre bundles. Ann. of Math. (2) **68** (1958), 348–371.

Let X, Y be algebraic varieties defined over an algebraically closed field, and let $\Phi: Y\to X$ be a regular map. If $\Phi(y)=x$, then Φ induces a homomorphism of local rings $\mathcal{O}_x\to\mathcal{O}_y$ and thus endows $\mathcal{O}_y$ with a module structure over $\mathcal{O}_x$. If now $\mathcal{F}$ is a coherent algebraic sheaf on X one may define a coherent algebraic sheaf $\mathcal{F}^\Phi$ on Y whose stalk at y is $\mathcal{F}_x\otimes_{\mathcal{O}_x}\mathcal{O}_y$. If $\mathcal{F}$ is a locally free sheaf, so that it is the sheaf of germs of regular sections of a vector bundle F, then $\mathcal{F}^\Phi$ is just the sheaf of germs of regular sections of the induced bundle $\Phi^{-1}(F)$. The main result of this paper can be stated as follows. If Y is a fibre bundle over X with a projective space as fibre, then we have a natural isomorphism: $H^q(X, \mathcal{F})\cong H^q(Y, \mathcal{F}^\Phi)$ for all $q\geq 0$.

The corresponding result in the complex analytic case (at least when $\mathcal{F}$ is locally free) has already appeared in the work of Bott [Ann. of Math. (2) **66** (1957), 203–248; MR **19**, 681] where it was deduced from an analytic Künneth formula of Grothendieck [Mem. Amer. Math. Soc., no **16** (1955); MR **16**, 618]. The proof in the algebraic case is formally similar though the corresponding Künneth formula is much more elementary.

If $\mathcal{F}$ is locally free, corresponding to a vector bundle F, and if Y is the projective bundle associated to the dual F^* of F, a more precise statement is possible. First we observe that $\Phi^{-1}(F^*)$ contains a sub-line-bundle Z^*; in fact the fibre Y_x is by definition the projective space associated to the vector space F_x^*, and the fibre Z_y^* is the one-dimensional subspace of F_x^* represented by the point y of Y_x. Dually therefore $\Phi^{-1}(F)$ has a line-bundle Z as a quotient. Let $\mathcal{Z}$ be the locally free sheaf on Y corresponding to Z. The authors prove that the homomorphism $\mathcal{F}^\Phi\to\mathcal{Z}$ induces isomorphism of all cohomology

groups. Combining this with the general result above one obtains the isomorphism: $H^q(X, \mathcal{F}) \cong H^q(Y, \mathcal{Z})$ for $q \geq 0$.

M. F. Atiyah (Cambridge, England)

LINEAR ALGEBRA

See also 1698.

1313:

★Mostowski, Andrzej; i Stark, Marceli. Algebra liniowa. [Linear algebra.] Biblioteka Matematyczna, Vol. 19. Państwowe Wydawnictwo Naukowe, Warsaw, 1958. 188 pp.

This book is considered as a supplementary volume to the authors' *Elements of higher algebra* [Vol. 1, Polskie Towarzystwo Matematyczne, Warsaw, 1953; Vol. 2, 3 Państwowe Wydawnictwo Naukowe, Warsaw, 1954; MR **15**, 594; MR **16**, 104, 663]. It deals with finite dimensional vector spaces and is written for the students who have studied theory of determinants and linear equations and possess some "mathematical maturity". In the first two chapters the fundamental concepts of a vector space and linear transformations are introduced. The headings of the next three chapters are: III. Structure of linear transformations; IV. Hermitian forms; V. Unitary spaces. The presentation may be said to be "classical": the discussion confines itself to the real and complex number fields and the concept of the group is not used. However, a very clear presentation, a careful attention to didactic aspects, instructive examples and satisfactory number of problems make this volume a valuable textbook for an introductory course in linear algebra.

J. W. Andrushkiw (Newark, N.J.)

1314:

Kawashima, Genkichi. Les générateurs génétiques des matrices stochastiques. Bull. Univ. Osaka Prefecture Ser. A **6** (1958), 25–32.

"The problem of constructing a sequence of stochastic matrices is studied by introducing a certain multiplication (which is called quasi-Jordan multiplication)." (Author's summary.)

The definition of this quasi-Jordan product is $a \times b = [\frac{1}{2}(ab+ba)]^{1/2}$. *O. Taussky-Todd* (Pasadena, Calif.)

1315:

Good, R. A. Systems of linear relations. SIAM Rev. **1** (1959), 1–31.

This paper is a modified version of notes based on six lectures presented by A. W. Tucker to the Summer Institute for College Mathematics Teachers, sponsored by the Social Science Research Council at Stanford University in 1957. The style is expository, therefore, and the subject matter is well-known. Two main topics form the central theme: the principle of duality and the nature of the set of solutions of a system of linear relations. The main tool used in the study, called the "Key Theorem", is the following: If A is any m by n real matrix and X and U are n-vectors, then the system $AX=0$, $X \geq 0$, $UA \geq 0$ possesses a solution X^* and U^* such that $X^*+U^*A>0$.

H. W. Kuhn (London)

1316:

Kato, Tosio. On positive eigenvectors of positive infinite matrices. Comm. Pure Appl. Math. **11** (1958), 573–586.

According to a classical theorem of Perron and Frobenius a finite positive matrix A has a positive dominant eigenvalue and a positive corresponding eigenvector. The author investigates infinite matrices from this point of view having the special structure $A=B-C$, where C is a small perturbation. The elements of B are given by

$$b_{ik} = f(i+\theta, k+\theta), \quad \theta > 0,$$

f being a function of two independent variables restricted by some conditions. The most interesting result of the paper may be stated as follows. There are real numbers θ_0, ω such that for $\theta \geq \theta_0$ every $\lambda > \omega$ is an eigenvalue of A and the associated eigenvector can be chosen positive if A is positive. The Hilbert matrix

$$b_{ik} = (i+k+2\theta)^{-1}$$

is a special case of the adopted type of matrices. As an application of his general theorems the author re-establishes a result of M. Rosenblum [Proc. Amer. Math. Soc. **9** (1958), 137–140; MR **20** #1139] concerning the Hilbert matrix. *E. Stiefel* (Zürich)

ASSOCIATIVE RINGS AND ALGEBRAS

1317:

Harada, Manabu. A note on Hattori's theorems. J. Inst. Polytech. Osaka City Univ. Ser. A **9** (1958), 43–45.

The author proves the following result which is a slight generalization of a result by A. Hattori [J. Math. Soc. Japan **9** (1957), 381–385; MR **20** #854b].

Let Λ be a commutative ring having no nilpotent element. Then the following four conditions are equivalent to each other: (a) Λ is semi-hereditary; (b) $\mathrm{Tor}_2{}^\Lambda(A, B)=0$ for any Λ-modules A and B; (c) $\mathrm{Tor}_1{}^\Lambda(X, A)=0$ for an arbitrary Λ-module X and for any torsion free Λ-module A; (d) $A \otimes_\Lambda B$ is torsion free for any torsion free Λ-modules A and B. Here the terminology "torsion free" is understood as follows: A Λ-module A is said to be torsion free if every non-zero-divisor in Λ is not a zero-divisor with respect to A.

M. Nagata (Kyoto)

1318:

Seshadri, C. S. Triviality of vector bundles over the affine space K^2. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 456–458.

To prove the triviality of vector bundles over the affine space K^n, it is sufficient (in fact, equivalent) to show that every finitely generated projective module over $K[X_1, \cdots, X_n]$ is free. In this note the author proves that if R is a principal ideal domain, then every finitely generated projective module over $R[X]$ is free. In particular, letting $R=K[X_1]$, he obtains the result that if P is a finitely generated projective $K[X_1, X_2]$-module, then P is free. The major step in the proof of this result is the following.

For any commutative ring R, let $S_n(R)$ be the $n \times n$ matrices over R with determinant 1. If R is a principal

ideal domain, and $\mathfrak{p}$ is a prime ideal of R, then the (natural) map of $S_n(R)$ into $S_n(R/\mathfrak{p})$ is onto.

D. Buchsbaum (Providence, R.I.)

1319:

Rosenberg, Alex. The structure of the infinite general linear group. Ann. of Math. (2) **68** (1958), 278–294.

Let $\mathfrak{M}$ be a vector space of dimension $\aleph_0$ over a division ring D. Let L denote the ring of all linear transformations of $\mathfrak{M}$, and G the group of invertible elements of L. This paper is concerned with determining the normal subgroups of G. For commutative D, and $\mathfrak{M}$ of finite dimension, the problem was studied by Jordan, Burnside and Dickson. For arbitrary D, and $3 \leq \dim \mathfrak{M} < \infty$, Dieudonné [Bull. Soc. Math. France **71** (1943), 27–45; MR **7**, 3] showed that all normal subgroups not in $Z^* \cdot 1$ contain the commutator subgroup G', and $G'/G' \cap Z^* \cdot 1$ is simple (Z^* is the set of non-zero elements of the centre Z of D). The principal tool was the proof that G' is generated by the transvections, those linear transformations of the form $1+F$, with F of rank 1, and $F^2=0$. In the present case ($\dim \mathfrak{M} = \aleph_0$) it is shown that $G'=G$ but the transvections no longer suffice to generate G. Theorem A states that G is generated (in the purely algebraic sense) by the elements of the form $1+C$ with $C^2=0$. Denote by T_ν the set of all linear transformations of rank $<\aleph_\nu$ ($\nu \leq \delta$), and by $G_{\nu(1)}$ the subgroup of G consisting of all elements of the form $1+F$, $F \in T_\nu$. Also, let G_ν be the subgroup of G of all elements of the form $z \cdot 1 + F$, with z in Z^* and F in T_ν. The main result (Theorem B) states that if N is a proper normal subgroup of G then either (a) $N \leq G_0$ or (b) $N = H \cdot 1 \times G_{\nu(1)}$ (direct product) where H is a subgroup of Z^*. In particular G_δ is a maximal normal subgroup so that G/G_δ is simple. The result applies to the groups G_ν as well, since it is shown that any normal subgroup of G_ν is a normal subgroup of G. In the final section the group U of invertible elements of the simple ring L/T_δ is studied. Let U' be the commutator subgroup of U. Theorem C states: (a) $U' = \bar{G}$, the image of G in L/T_δ; (b) $U'/Z^* \cdot 1 = U'/U' \cap Z$ is simple; (c) if $\delta > 0$, $U'=U$; if $\delta = 0$, U/U' is infinite cyclic. *K. G. Wolfson* (New Brunswick, N.J.)

1320:

Deskins, W. E. A note on the system generated by a set of endomorphisms of a group. Michigan Math. J. **6** (1959), 45–49.

Let E be a set of endomorphisms of a group G satisfying the descending chain condition on E-subgroups. Now extend E to the semigroup E' of all products of finitely many elements of E. The set $R(E)$ consisting of elements of the form $\sum r_i E_i$, $E_i \in E'$, r_i rational integers, is a near-ring satisfying: (i) $R(E)$ contains a multiplicative semigroup D of right distributive elements d such that each element of $R(E)$ can be written as a finite linear combination $\sum r_i d_i$ of $d_i \in D$ with rational integral coefficients r_i; (ii) $R(E)$ satisfies the descending chain condition for right modules.

The author now defines any near-ring satisfying (i) and (ii) as a distributively-generated near-ring N (DGN-ring) and develops a structure theory for DGN-rings of radical, semisimplicity and simplicity. Using standard algebraic arguments the results are not surprising and the following are typical: $\bar{N} = N - R$ (R the radical) is semisimple and expressible uniquely as a direct sum of simple DGN-rings N_i,

$$\bar{N} = N_1 \oplus \cdots \oplus N_r.$$

Each N_i has a left identity e_i and $e = e_i + \cdots + e_r$ is the left identity of $\bar{N}$.

A simple DGN-ring N with unity element $e = \sum e_i$ (where e_i are mutually orthogonal idempotents) is a ring if and only if each $e_i N e_j$ is a commutative group relative to addition. *L. J. Paige* (Los Angeles, Calif.)

1321:

Kertész, A. Beiträge zur Theorie der Operatormoduln. Acta Math. Acad. Sci. Hungar. **8** (1957), 235–257.

This paper contains a number of characterisations of completely reducible modules, a proof that a ring over which every module is a direct sum of a trivial module and a completely reducible module is semisimple in the classical sense, and a generalisation to submodules of the "serving subgroups" of abelian group theory.

Graham Higman (Oxford)

1322:

Leavitt, W. G. Note on two problems of A. Kertész. Publ. Math. Debrecen **6** (1959), 83–85.

There are examples constructed (namely certain rings of non-commutative polynomials) which answer negatively the following questions raised by the reviewer. 1. If R is an arbitrary (associative) ring and the order of each element ($\neq 0$) of an R-module G is the intersection of a finite number of maximal left ideals of R, does then $RG=G$ necessarily hold [same Publ. **4** (1956), 229–236; MR **18**, 108]? 2. If L is a maximal left ideal of R and $R^2 \not\subset L$ ($R^2 \neq L$), is then L regular (in other terminology: modular)? In connection with the second problem it is proved that a maximal left ideal L of an arbitrary ring is regular if and only if the ring contains an element which maps the complement L' of L into itself under right multiplication. *A. Kertész* (Debrecen)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

See also 1352.

1323:

Širšov, A. I. Some questions in the theory of rings close to associative. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 3–20. (Russian)

This is a survey of the development of the theory of alternative, Jordan and Lie rings and their nearest generalizations including power-associative rings. In each case unsolved problems and the topics next awaiting investigation are noted. The bibliography lists 74 items, mostly dated 1950–1958.

I. M. H. Etherington (Edinburgh)

1324:

Jacobson, N. Nilpotent elements in semi-simple Jordan algebras. Math. Ann. **136** (1958), 375–386.

Let J be a finite-dimensional semisimple Jordan algebra over a field Φ of characteristic $\neq 2$, 3, and let e be a nilpotent element of J. The author proves Theorem A: There exists a second nilpotent element f in J such that the subalgebra K generated by e and f is a direct sum of

ideals which are isomorphic to the Jordan algebra of all linear transformations of a finite-dimensional vector space M over Φ which are self-adjoint relative to a non-degenerate symmetric bilinear form (x, y) of maximal Witt index. The proof of Theorem A for characteristic 0 is based on Theorem B: There exists a second nilpotent element f in J such that $2A(e, e, f)=e$, $2A(f, f, e)=f$, where $A(x, y, z)=(xy)z-x(yz)$. The author has proved Theorem B for characteristic 0 in an earlier paper [Proc. Amer. Math. Soc. **2** (1951), 105–113; MR **14**, 241]. The proofs of both theorems for characteristic $\neq 0$ rely on the known structure theory for semisimple Jordan algebras.

Any nonzero nilpotent element of an exceptional simple Jordan algebra J has index either three or two. The results above are applied to prove that any two nilpotent elements of index three are conjugate relative to automorphisms of J and, if Φ is algebraically closed, that the same result holds for index two. *R. D. Schafer* (Cambridge, Mass.)

1325:

Iwahori, Nagayoshi. On real irreducible representations of Lie algebras. Nagoya Math. J. **14** (1959), 59–83.

The problem of finding all real irreducible representations of a given real Lie algebra $\mathfrak{g}$ is solved. First it is reduced to the problem of finding all complex irreducible representations (ρ, V) of $\mathfrak{g}$ and of deciding whether (ρ_R, V_R) is irreducible or not. Here (ρ_R, V_R) is the real representation obtained from (ρ, V) by regarding V as a real vector space. Secondly the problem is reduced to the case where $\mathfrak{g}$ is simple, and finally the solution is formulated in terms of a set of fundamental weights of $\mathfrak{g}^C$ (complexification of $\mathfrak{g}$) and the highest weight of (ρ, V).

R. Ree (Kingston, Ont.)

1326:

Boers, A. H. Le n-associateur dans un anneau commutatif. Nederl. Akad. Wetensch. Proc. Ser. A. **62** = Indag. Math. **21** (1959), 39–44.

The author applies his concept of n-associator [*Généralisation de l'associateur*, J. Van Tuyl, Antwerpen-Zaltbommel, 1957; MR **19**, 9] to commutative rings. A typical theorem states that a $2n$-alternative commutative ring of characteristic $\neq 2$ is $2n$-associative.

R. D. Schafer (Cambridge, Mass.)

1327:

Jenner, W. E. A note on truncated loop algebras. Portugal. Math. **16** (1957), 1–2.

The reviewer [R. H. Bruck, Trans. Amer. Math. Soc. **56** (1944), 141–199; MR **6**, 116] defined the class of "truncated loop algebras", showed that some algebras of the class are (1) both left and right simple, and asked whether (2) an algebra isotopic to a truncated loop algebra is necessarily simple. The author shows that (1) holds for every truncated loop algebra and constructs a counter-example to (2). *R. H. Bruck* (Madison, Wis.)

HOMOLOGICAL ALGEBRA

See also 1317.

1328:

Grothendieck, Alexandre. Sur quelques points d'algèbre homologique. Tôhoku Math. J. (2) **9** (1957), 119–221.

The formal analogy between the cohomology theory of a space with coefficients in a sheaf, and the derived functors of functors of modules has been apparent for some time. The author has here developed the necessary framework to encompass these (as well as other) theories.

To attain the generality required for this unification, categories are introduced and studied in some detail (Chapter I). Although abelian categories [exact categories in the sense of Buchsbaum, Trans. Amer. Math. Soc. **80** (1955), 1–34; MR **17**, 579] play the fundamental role in this work, the author introduces the notion of additive category which is useful in speaking about spectral functors. (An additive category is given by a collection of objects $\{A\}$, a collection of abelian groups $\{H(A, B)\}$ for every pair of objects (A, B), and a law of composition $H(A, B) \otimes H(B, C) \to H(A, C)$. These terms are required to satisfy some axioms which essentially give the associativity of composition, the existence of an identity in $H(A, A)$ for each A, and the existence of a zero object. Finite direct sums are also postulated to exist. Abelian categories, then, are additive categories in which the existence of kernel, image, cokernel, and coimage is postulated for every map.) The main result of the first chapter is that if an abelian category satisfies two additional conditions (essentially that there exist an object in the category which "separates points" (a so-called generator), and that direct limits preserve some reasonable inclusion relations (axiom AB5)), then every object can be imbedded in an injective object.

This result enables the author to develop, in Chapter II, the theory of derived functors of a given functor. The treatment in the first three sections of this chapter is very similar to that given by H. Cartan and S. Eilenberg in *Homological algebra*, Princeton Univ. Press, Princeton, N.J., 1956 [MR **17**, 1040] (the Cartan-Eilenberg terminology of "connected sequence of functors" is replaced here by "∂-functor"). The major contribution of Chapter II is a discussion of spectral sequences and spectral functors together with a theorem which gives a mechanical way of obtaining most of the well-known spectral sequences. This theorem states that if $F: C \to C'$, $G: C' \to C''$ are functors on abelian categories satisfying certain acyclicity conditions, then there is a spectral functor (i.e., a functor which to each object assigns a spectral sequence) whose E_2 term is $E_2^{p,q}(A) = R^pG(R^qF(A))$ and whose E_∞ term is the graded group associated with $R(GF)$ (suitably filtered). (If T is any functor, RT denotes the right derived functor of T, while R^nT denotes the right derived of degree n.)

In Chapter III, the cohomology of a space X with coefficients in a sheaf of abelian groups F with respect to a family of closed sets Φ, denoted by $H_\Phi(X, F)$, is defined as the right derived functor of $\Gamma_\Phi(F)$, where $\Gamma_\Phi(F)$ denotes the sections of F with support in Φ. The author shows that the category of sheaves of abelian groups satisfies the necessary conditions, stated in Chapter I, to insure that every sheaf can be imbedded in an injective one; thus the definition of these cohomology groups as derived functors makes sense. The Leray spectral sequence of a continuous map and a comparison with the Čech cohomology groups are obtained here, and under suitable conditions (e.g., when X is paracompact) the Čech cohomology groups are the same as those defined by derived functors. The introduction of soft and flabby sheaves (replacing the fine sheaves of yesteryear) facilitates the presentation of the theory.

In Chapter IV, the functors Ext and $\mathscr{E}xt$ are defined for sheaves of modules. Ext is defined as the right derived functor of Hom, while $\mathscr{E}xt$ is the right derived functor of $\mathscr{H}om$ (= sheaf of germs of sheaf homomorphisms). For a fixed sheaf of modules A, there is a spectral sequence relating $H^p(X; \mathscr{E}xt^q(A, B))$ with Ext (A, B).

In Chapter V, the situation is essentially the same as in IV, except that here a group G operates on the space X, on the sheaf of rings $\mathcal{O}$, and on the sheaves of $\mathcal{O}$-modules. The (non-topological) group G is allowed to operate with fixed points on X. Cohomology groups $H^n(X; G, A)$ are defined, and again the main theorems have to do with the existence of certain interesting spectral sequences.

The lack of mention of specific theorems in the paragraphs above is due not to the paucity of results in this paper but to the fact that the statements of these results would require too much preliminary definition.

D. Buchsbaum (Providence, R.I.)

1329:

Nunke, R. J. Modules of extensions over Dedekind rings. Illinois J. Math. **3** (1959), 222–241.

This paper studies the structure of the R-module $\mathrm{Ext}_R{}^1(A, C)$, where R is a Dedekind ring and A, C are R-modules. Let $(e): 0 \to C \to E \to A \to 0$ be an extension of C by A, and let I be a non-zero ideal of R. Then the characteristic class $\chi(e)$ lies in $I \cdot \mathrm{Ext}_R{}^1(A, C)$ if and only if $JC = C \cap JE$ for every ideal J containing I. Let $R^\omega \, \mathrm{Ext}_R{}^1(A, C)$ denote the intersection of all the modules $I \cdot \mathrm{Ext}_R{}^1(A, C)$ with I ranging over the non-zero ideals of R. Then (e) is a pure extension if and only if $\chi(e)$ is in $R^\omega \, \mathrm{Ext}_R{}^1(A, C)$. It is shown that mod its submodule of divisible elements, $\mathrm{Ext}_R{}^1(A, C)$ is a direct product of its P-adic completions, P ranging over the prime ideals of R. It follows that a module M which is either torsion-free or torsion is isomorphic to $\mathrm{Ext}_R{}^1(A, C)$ for some modules A and C if and only if it is the direct product of its P-adic completions. In particular R itself has this property if and only if it is a complete discrete valuation ring. Among other things some results of R. Baer are generalized. First, if C is a torsion module, then $\mathrm{Ext}_R{}^1(A, C) = 0$ for every torsion-free R-module A if and only if C is the direct sum of a divisible module and a module with bounded order. Second, if A is any module such that $\mathrm{Ext}_R{}^1(A, C) = 0$ for every torsion module C, then A is torsion-free and every submodule of A with countable rank is projective.

E. Matlis (Evanston, Ill.)

GROUPS AND GENERALIZATIONS

See also 1319.

1330:

Chen, K. T.; Fox, R. H.; and Lyndon, R. C. Free differential calculus. IV. The quotient groups of the lower central series. Ann. of Math. (2) **68** (1958), 81–95.

Parts I, II, III of the paper of like title are by Fox [Ann. of Math. (2) **57** (1953), 547–560; **59** (1954), 196–210; **64** (1956), 407–419; MR **14**, 843; **15**, 931; **20** #2374].

The quotient groups $Q_n(G) = G_n/G_{n+1}$ of the lower central series $G = G_1 \supset G_2 \supset \cdots$ of a finitely generated group G are finitely generated abelian groups. In § 4 the authors give an algorithm which (in principle, at least) allows the calculation of Q_n from any given finite presentation of G. The details depend upon the free differential calculus (see I, II, III above) and will be omitted. §§ 1, 2, 3 of the paper are in a sense preparatory to § 4. Nevertheless these sections have an interest in their own right, in that they give an alternative and highly satisfactory derivation of known results concerning free groups; and they will be discussed here at some length.

In § 1 the authors suppose given an ordered set (finite or infinite, and not necessarily well-ordered) which they denote by $1, 2, \cdots, q$. The free semigroup $\mathfrak{A}$ in the set is the set of all ordered sequences a formed from $1, 2, \cdots, q$, of finite positive length $n(a)$, with juxtaposition as multiplication. $\mathfrak{A}_1$ is the given set, $\mathfrak{A}_n$ is the set of all sequences of length n. They order $\mathfrak{A}$ lexicographically and give four inductive definitions (later proved to be equivalent) of a standard sequence. All four definitions are needed for the proofs; nevertheless, we mention only one: the sequence c is in $\mathfrak{A}'$ (the set of standard sequences) if either c is in $\mathfrak{A}_1$ or $c = ab$ where a, b are in $\mathfrak{A}'$ and $a < b$. For each n, $\mathfrak{A}_n'$ denotes the set of standard sequences of length n. The authors show that each c in $\mathfrak{A}_n$, $n \geq 2$, has a unique "standard representation" $c = ab$ where a is in $\mathfrak{A}$, b is in $\mathfrak{A}'$, and b has maximal length; if c is itself in $\mathfrak{A}'$, so is a. For the case that the order, q, of the generating set is finite, they compute the order, $\psi_n(q)$, of $\mathfrak{A}_n'$. This is the Witt number [E. Witt, J. Reine Angew. Math. **177** (1937), 152–160].

In § 2 the set $\mathfrak{M}$ of monomials is defined inductively in terms of the sets $\mathfrak{M}_n$ of monomials of length n. Here $\mathfrak{M}_1 = \mathfrak{A}_1$. For $n > 1$, the elements of $\mathfrak{M}_n$ are the symbols (a, b) where a is in $\mathfrak{M}_r$, b is in $\mathfrak{M}_s$, with $n = r + s$ and, in case $r = s$, $a \neq b$. The set $\mathfrak{M}'$ of standard monomials is, by definition, the image of $\mathfrak{A}'$ under the following mapping $a \to a^\times$: If k is in $\mathfrak{A}_1' = \mathfrak{A}_1 = \mathfrak{M}_1$, $k^\times = k$. If $n \geq 2$ and if c, in $\mathfrak{A}_n'$, has standard representation $c = ab$, then $c^\times = (a^\times, b^\times)$. —Thus, for example, $(123)^\times = (1, (2, 3))$.—The image of $\mathfrak{A}_n'$ is the set $\mathfrak{M}_n'$ of standard monomials of weight n.

One of the authors' results in § 3 could be interpreted as stating that, for the free group X on generators $x_1, \cdots, x_q$, $Q_n(X)$ is isomorphic to the free abelian group with the set $\mathfrak{M}_n'$ of standard monomials (or the set $\mathfrak{A}_n'$ of standard sequences) of length n as a free set of generators. (In particular, then, for q finite, $\psi_n(q)$ is the rank of $Q_n(X)$; moreover, the authors' description of standard monomials is an alternative to the basic commutators of M. Hall [Proc. Amer. Math. Soc. **1** (1950), 575–581; MR **12**, 388].) Actually the authors proceed a little differently. Let Y_n be the free (non-abelian!) group generated by the set $\mathfrak{M}_n$ of (all) monomials of weight n. A homomorphism $\lambda(n)$ of Y_n into X_n is defined inductively by: $k^{\lambda(1)} = x_k$ if k is in $\mathfrak{M}_1 = \mathfrak{A}_1$; $(a, b)^{\lambda(n)} = [a^{\lambda(r)}, b^{\lambda(s)}]$ if a is in $\mathfrak{M}_r$, b is in $\mathfrak{M}_s$ and $r + s = n$. (Here $[x, y]$ denotes a group commutator.) In particular, the image of $\mathfrak{M}_n'$ under $\lambda(n)$ is defined to be the set of all standard commutators of weight n.

Let JX be the group ring of X over the ring J of integers. With each a in $\mathfrak{A}$ the authors associate a "free derivative" $D_a{}^0$ (mapping JX into itself). It turns out that, for each a in $\mathfrak{A}_n$, $D_a{}^0$ induces a homomorphism of X_n upon an additive abelian group and of X_{n+1} upon 0. Moreover, if c is in $\mathfrak{A}_n'$, and if $w = (c^\times)^{\lambda(n)}$ is the corresponding standard commutator, $D_c{}^0$ maps w upon 1, whereas, for each a in $\mathfrak{A}_n$ such that $a < c$, $D_a{}^0$ maps w upon 0. With these results, it is easy to prove that the standard commutators of weight n form a free basis for X_n modulo

X_{n+1}. The authors go on to determine a number of relations needed for the algorithm in § 4.

R. H. Bruck (Madison, Wis.)

1331:

Hall, P. Some word-problems. J. London Math. Soc. **33** (1958), 482–496.

This paper is a presidential address delivered to the London Mathematical Society on November 21, 1957. Written, presumably, for the non-specialist, it gives a clear discussion of certain word problems but omits all references to the relevant literature, aside from the names of the chief authors.

Preliminary topics: words, algebraic operations, the concept of a word problem, varieties of algebras, laws. Brief mention: results on the word problems for groups, semigroups, cancellation semigroups. Detailed discussion: formation of words without brackets ("the paradox of the pointlessness of punctuation"); normal forms, illustrated by the solution of the word problem for Lie rings and that for nilpotent groups, each in terms of "basic products". Brief concluding remarks: the state of the Burnside problems for groups; unsolved word problems connected with the Burnside problems and with the theory of Engel groups.

We quote the last two sentences of the paper: "Problems such as these still seem to present a formidable challenge to the ingenuity of algebraists. In spite of, or perhaps because of, their relatively concrete and particular character, they appear, to me at least, to offer an amiable alternative to the ever popular pursuit of abstractions."

R. H. Bruck (Madison, Wis.)

1332:

Grün, Otto. Automorphismen von Gruppen und Endoisomorphismen freier Gruppen. Illinois J. Math. **2** (1958), 759–763.

Let $\mathfrak{F}$ be a finitely generated free group, $\mathfrak{N}$ a normal subgroup, $\mathfrak{G}=\mathfrak{F}/\mathfrak{N}$. Assume that $\mathfrak{G}$ is not isomorphic with a proper factor group of itself. Then any isomorphism σ of $\mathfrak{F}$ into itself with $\mathfrak{F}^\sigma\mathfrak{N}=\mathfrak{F}$ and $\mathfrak{N}^\sigma\subseteq\mathfrak{N}$ induces an automorphism of $\mathfrak{G}$. The author states that every automorphism of $\mathfrak{G}$ can be obtained in this way, and sketches a proof. In case $\mathfrak{N}$ is the subgroup generated by all mth powers, the author constructs an Abelian group $\mathfrak{A}$ of automorphisms of the Burnside group $\mathfrak{G}$. $\mathfrak{A}$ has order $\phi(m)^\alpha$ (α is the number of generators) and contains only 2^α elements which are induced by automorphisms of $\mathfrak{F}$.

P. J. Higgins (London)

1333:

Rühs, Fritz. Über das allgemeine Rédeische schiefe Produkt. J. Reine Angew. Math. **200** (1958), 99–111.

Rédei's skew product [same J. 188 (1950), 201–227; MR 14, 13] $G \circ \Gamma$ of two groups G and Γ consists of the pairs (a, α) with $a \in G$, $\alpha \in \Gamma$ and the multiplication rule $(a, \alpha)(b, \beta)=(ab^\alpha\beta^a, a^\beta\alpha^b\beta)$, where b^α, $\beta^a \in G$, a^β, $\alpha^b \in \Gamma$ are functions of the stated arguments. A skew product is called k-fold degenerate if k of the following relations hold identically: $b^\alpha=b$, $\beta^a=1$, $a^\beta=1$, $\alpha^b=\alpha$. The 4-fold degenerate product is, of course, the direct product; the 3-fold degenerate products are very easy to describe. The two-fold degenerate products have been investigated by Kochendörffer [same J. **192** (1953), 96–101; MR **15**, 852] and the simply degenerate products by the present author [same J. **198** (1957), 81–86; MR **20** ~~#[illegible]~~ #900]. In all these cases it turns out that the skew product is a single or a repeated Schreier extension of and by certain subgroups, or by a so-called Zappa-Szép product. In the present paper the author deals with the most general, the non-degenerate case. The results are not complete, but at least he obtains a number of sufficiency theorems of the type: if certain subgroups satisfy certain conditions, then the non-degenerate skew product is a Schreier extension of a certain subgroup by a certain Zappa-Szép product. In particular, this is always the case when G and Γ are abelian or hamiltonian. The detailed conditions are very elaborate and cannot be reproduced here.

K. A. Hirsch (London)

1334:

Marchionna Tibiletti, Cesarina. Sul prodotto di gruppi permutabili. Ann. Mat. Pura Appl. (4) **43** (1957), 341–356.

Soient Γ_1, Γ_2 deux groupes de permutations de supports respectifs M_1, M_2 (le mot "permutation" est ici synonyme de surjection biunivoque, sans aucune condition de finitude). Le produit complet $\Gamma_1 \circ \Gamma_2$ des Γ_1, Γ_2 est le groupe de toutes les permutations de $M=M_1\times M_2$ de la forme $\sigma=[\sigma_1, \sigma_2(t_1)]$ $(\sigma_1 \in \Gamma_1, \sigma_2(t_1) \in \Gamma_2{}^{M_1})$, où la permutation ainsi notée opère dans M selon la règle

$$[\sigma_1, \sigma_2(t_1)]\cdot(x_1, x_2) = (\sigma_1\cdot x_1, \sigma_2(x_1)\cdot x_2).$$

La loi de composition de $\Gamma_1 \circ \Gamma_2$, considéré comme un groupe abstrait, est

$$[\sigma_1, \sigma_2(t_1)][\tau_1, \tau_2(t_1)] = [\sigma_1\tau_1, \sigma_2(\tau_1\cdot t_1)\tau_2(t_1)].$$

M. Krasner et L. Kaloujnine [Acta Sci. Math. Szeged **13** (1950), 208–230; **14** (1951), 39–82; MR **14**, 242] ont montré que $\Gamma_1 \circ \Gamma_2$ est un groupe universel pour les groupes satisfaisant à certaines conditions formulables à l'aide des Γ_1 et Γ_2. Un sous-groupe transitif de $\Gamma_1 \circ \Gamma_2$ est dit schreierien s'il est régulier (notion considérée par Krasner et Kaloujnine seulement si Γ_1 et Γ_2 sont réguliers). Soient donnés deux groupes abstraits A, B, deux sous-groupes $I_A\subseteq A$ et $I_B\subseteq B$ isomorphes l'un à l'autre et un isomorphisme ω de I_A sur I_B. L'auteur réussit à caractériser (à l'isomorphie près), à l'aide des produits complets convenables, les groupes G possédant des sous-groupes permutables A^*, B^* isomorphes respectivement à A, B et tels que: (1) $G=A^*B^*$; (2) $I^*=A^*\cap B^*$ est isomorphe à I_A (et à $I_B=\omega\cdot I_A$) et on peut choisir des isomorphismes $\omega_A{}^*$ de A^* sur A et $\omega_B{}^*$ de B^* sur B de manière qu'on ait $\omega\,\omega_A{}^*=\omega_B{}^*$ sur I^*. Voici la solution de ce problème: soient M_A l'ensemble des classes αI_A $(\alpha\in A)$ dans A (mod I_A) et A/I_A le quotient permutationnel de A par I_A, c'est-à-dire le groupe des permutations de M_A induites par A. Soit $S(M)$ le groupe symétrique de support M (c'est-à-dire celui de toutes les permutations de M). En vertu de la théorie de Krasner et Kaloujnine, A peut se représenter (d'une manière parfaitement déterminée par un choix des représentants $\rho(\bar{x})$ des $\bar{x}$ parcourant M_A) comme un sous-groupe de $(A/I_A)\circ I_A$. Or, ω induit l'isomorphisme $[\alpha, \beta(t_1)]\to[\alpha, \omega\cdot\beta(t_1)]$ $(\alpha\in A/I_A; \beta(t_1)\in I_A{}^{M_A})$ de $(A/I_A)\circ I_A$ sur $(A/I_A)\circ I_B\subseteq S(M_A)\circ B$. Ainsi, si l'on fixe une représentation de A dans $(A/I_A)\circ I_A$, on obtient aussi un sous-groupe $\tilde{A}$ de $S(M_A)\circ B$ représentant A. L'auteur montre que les groupes G considérés coïncident, à l'isomorphie près, avec les sous-groupes schreieriens de

$S(M_A) \circ B$ contenant $\bar{A}$. En particulier, si I_A (et I_B) se réduit à l'unité, on a $M_A = A$ et la représentation $A \to \bar{A}$ est $\alpha \to (\alpha, e_B)$ où e_B est l'unité de B. Ainsi, dans ce cas, les groupes G sont les sous-groupes schreieriens de $S(A) \circ B$ contenant (A, e_B). Le travail contient encore la caractérisation de certains groupes plus généraux que les précédents, qui se trouvent coïncider avec les sous-groupes arbitraires de $S(M_A) \circ B$ contenant $\bar{A}$.

M. Krasner (Paris)

1335:

Marchionna Tibiletti, Cesarina. Sui prodotti completi contenenti prodotti di gruppi permutabili. Ann. Mat. Pura Appl. (4) **44** (1957), 153–170.

Il s'agit du même problème que dans le travail précédent de l'auteur [#1334], qui cherche s'il existe quelque sous-groupe propre Δ de $S(M_A)$ tel que les groupes G de la forme considérée soient déjà tous, à l'isomorphie près, des sous-groupes de $\Delta \circ B$ contenant $\bar{A}$. Soit Δ le groupe de permutations de M_A engendré par A/I_A et par tous les groupes Σ tels que: (1) Σ est une image homomorphe de B; (2) Σ est permutable avec A/I_A; (3) $\Sigma \cap (A/I_A)$ coïncide avec le groupe de stabilité de l'élément $m = I_A$ de M_A dans A/I_A; (4) tous les éléments de Σ laissent invariant m. Alors, l'auteur prouve que, dans la solution du problème considéré, $S(M_A) \circ B$ peut se remplacer par $\Delta \circ B$. L'auteur donne quelques exemples, où Δ est un sous-groupe propre de $S(M_A)$. Par ex., dans le cas où I_A se réduit à l'unité, on a, quand la représentation régulière de A est contenue dans le groupe alternatif D_A des permutations de A et quand B est simple, $\Delta \leq D_A$. D'autre part, l'auteur donne des exemples, où $S(M_A) \circ B$ ne peut pas se remplacer par aucun groupe plus petit.

{Remarque du référent: Δ peut se remplacer par le groupe Δ', en général plus petit, qui est le composé de A/I_A et de tous les sous-groupes Σ' de $S(M_A)$, satisfaisant aux mêmes conditions (1), (2) et (4) que les Σ et à la condition: (3′) l'homomorphisme ρ de B sur Σ' peut être choisi de telle manière que, pour tout $i \in I_A$, $\rho\omega \cdot i$ coïncide avec l'image de i dans A/I_A (cette condition est plus forte que (3)).}

M. Krasner (Paris)

1336:

Marchionna Tibiletti, Cesarina. Immersione in prodotti completi di prodotti ordinati di più gruppi. Ann. Mat. Pura Appl. (4) **44** (1957), 233–244.

Le produit complet $\Gamma_1 \circ \Gamma_2 \circ \cdots \circ \Gamma_s$ des groupes de permutations $\Gamma_1, \Gamma_2, \cdots, \Gamma_s$ des ensembles respectifs $M_1, M_2, \cdots, M_s$ est l'ensemble des permutations de $M = M_1 \times M_2 \times \cdots \times M_s$ notées par les "tableaux"

$$\sigma = [\sigma_1, \sigma_2(t_1), \cdots, \sigma_i(t_1, t_2, \cdots, t_{i-1}), \cdots]$$

$$(\sigma_1 \in \Gamma_1, \sigma_2(t_1) \in {\Gamma_2}^{M_1}, \cdots, \sigma_i(t_1, \cdots, t_{i-1}) \in {\Gamma_i}^{M_1 \times \cdots \times M_{i-1}})$$

où un tel σ opère sur M selon la règle $\sigma \cdot (x_1, x_2, \cdots, x_i, \cdots) = (\sigma_1 \cdot x_1, \sigma_2(x_1) \cdot x_2, \cdots, \sigma_i(x_1, x_2, \cdots, x_{i-1}) \cdot x_i, \cdots)$, ce qui donne la loi de composition

$$\begin{aligned}&[\sigma_1, \sigma_2(t_1), \cdots, \sigma_i(t_1, \cdots, t_{i-1}), \cdots]\\ &\qquad \times [\tau_1, \tau_2(t_1), \cdots, \tau_i(t_1, \cdots, t_{i-1}), \cdots]\\ &= [\sigma_1\tau_1, \sigma_2(\tau_1 \cdot t_1)\tau_2(t_1), \cdots,\\ &\qquad \sigma_i(\tau_1 \cdot t_1, \cdots, \tau_{i-1}(t_1, \cdots, t_{i-2}) \cdot t_{i-1})\tau_i(t_1, \cdots, t_{i-1}), \cdots].\end{aligned}$$

Krasner et Kaloujnine ont montré [Acta Sci. Math. Szeged **13** (1950), 208–230; **14** (1951), 39–82; MR **14**, 242] que $\Gamma_1 \circ \Gamma_2 \circ \cdots \circ \Gamma_s$ est un groupe universel pour les groupes d'une certaine forme. Un sous-groupe transitif de $\Gamma_1 \circ \Gamma_2 \circ \cdots \circ \Gamma_s$ est dit schreierien s'il est régulier.

$A_1, A_2, \cdots, A_s$ étant des groupes abstraits, l'objet du travail est la caractérisation (à l'isomorphie près) des groupes (que l'auteur appelle produits ordonnés des $A_1, A_2, \cdots, A_s$) $G = A_1 A_2 \cdots A_s$ tels que, pour tout $j = 2, \cdots, s$, $G_j = A_j A_{j+1} \cdots A_s$ soit permutable avec A_{j-1} et que $G_j \cap A_{j-1}$ soit l'unité e de G. Le produit ordonné est dit à facteurs permutables si, pour tous i, j, on a $A_i A_j = A_j A_i$, et il est dit à chaîne normale ou principale si $G = G_1 \supset G_2 \supset \cdots \supset G_s \supset \{e\}$ est une telle chaîne. Posons $M_i = A_i$, $\Gamma_i = S(A_i)$ si $i < s$ et $\Gamma_s = A_s$. Si e_i est l'unité de A_i, appelons E_i l'ensemble des vecteurs $(x_1, x_2, \cdots, x_s) \in M$ tels que $x_j = e_j$ si $j \neq i$, et G étant un sous-groupe transitif de $S(A_1) \circ S(A_2) \circ \cdots \circ S(A_{s-1}) \circ A_s$, appelons $\bar{G}^{(i)}$ le sous-groupe de G préservant l'ensemble E_i (mais pouvant permuter ses éléments). L'auteur montre que les produits ordonnés des $A_1, A_2, \cdots, A_s$ se caractérisent (à l'isomorphie près) comme les sous-groupes G de $S(A_1) \circ S(A_2) \circ \cdots \circ S(A_{s-1}) \circ A_s$ tels que $\bar{G}^{(i)}$, pour tout $i = 1, 2, \cdots, s-1$, soit isomorphe à A_i. Si, en plus, tout élément de $\bar{G}^{(i)}$ préserve, pour tout $j \neq i$, la j-ième coordonnée de tout $(x_1, x_2, \cdots, x_s) \in M$ quand cette coordonnée est e_j, le produit ordonné G est à facteurs permutables. Enfin (comme cela résulte directement des résultats de Krasner et Kaloujnine), ce produit est à chaîne normale s'il est $\subseteq A_1 \circ A_2 \circ \cdots \circ A_{s-1} \circ A_s$, et à chaîne principale si, pour tout i, tout élément de $G^{(i)}$ préserve $e^{i-1} = (e_1, e_2, \cdots, e_{i-1})$, c'est-à-dire transforme tout $(e_1, e_2, \cdots, e_{i-1}, x_i, \cdots, x_s) \in M$ en un élément commençant également par

$$(e_1, e_2, \cdots, e_{i-1}, \cdots).$$

M. Krasner (Paris)

1337:

Marchionna Tibiletti, Cesarina. Sui minimi prodotti completi contenenti prodotti di gruppi permutabili. Ann. Mat. Pura Appl. (4) **44** (1957), 251–259.

L'auteur se pose la question si le groupe Δ défini dans son travail précédent [#1336] est le sous-groupe minimal de $S(M_A)$ tel que son produit complet par B donne la solution du problème de construction des groupes $G = AB$ avec A, B permutables et tels que $A \cap B = I$. Elle commence par déterminer le groupe Δ à partir des chaînes normales de $S(M_A)$ et de B, et montre ensuite que, sous certaines conditions, Δ est effectivement minimal. (Par exemple, quand: (1) tous les Σ sont isomorphes à B, ou (2) A/I_A est une représentation fidèle de A et appartient au normalisateur de tous les Σ.)

{Remarque du référent: on peut montrer que le groupe minimal cherché est Δ' et qu'ainsi la condition de minimalité de Δ est $\Delta = \Delta'$.}

L'ensemble de ces 4 travaux constitue une contribution intéressante à la théorie de l'extension des groupes. Il montre, en même temps, la puissance de la méthode de produit complet, qui permet de résoudre (du moins en principe) les problèmes qui étaient inabordables par d'autres méthodes (y compris la méthode cohomologique, qui n'est vraiment efficace que dans le cas, où quelque chose est abélien).

M. Krasner (Paris)

1338:

Peremans, W. Completeness of holomorphs. Nederl. Akad. Wetensch. Proc. Ser. A **60** = Indag. Math. **19** (1957), 608–619.

A group is complete it if has no centre and no outer automorphisms. It is well known that the holomorph of a non-Abelian group cannot be complete. For an Abelian group G in which the mapping $x \to 2x$ is an automorphism, the author gives necessary and sufficient conditions that the holomorph of G be complete. In particular, these conditions are satisfied if G is (i) indecomposable, (ii) a direct sum of cyclic groups, or (iii) divisible.

P. J. Higgins (London)

1339:

Wos, Lawrence Thomas. On commutative prime power subgroups of the norm. Illinois J. Math. **2** (1958), 271–284.

Let P be an (additively written) abelian p-group and let S be a multiplicative group of operators on P. In effect, the author defines the pair P, S to be normlike if, for each a in S, the operator $1-a$ (i) maps P into a cyclic subgroup and (ii) satisfies $(1-a)^2=0$. For such a pair it is shown that $S=Z_2(S)$ and $P=F_3$. Here $F_0=0$ and, for i positive, F_i is the set of all x in P such that $x(1-a)\equiv 0 \bmod F_{i-1}$ for every a in S. For the case that p is odd, S is abelian precisely when $P=F_2$.

Next suppose that P is an abelian normal p-subgroup of a group G and that S is the group of automorphisms of P induced by the inner automorphisms of G. Such a pair is called a norm pair provided P is contained in the norm, $N(G)$, namely, the intersection of the normalizers of the subgroups of G. Norm pairs need not be normlike, but the connection is so close that the author proves the following: $N(G)$ is contained in $Z_3(G)$. This sharpens a result of Baer [Publ. Math. Debrecen **4** (1956), 347–350; MR **18**, 109].

R. H. Bruck (Madison, Wis.)

1340:

Conrad, Paul. The group of order preserving automorphisms of an ordered abelian group. Proc. Amer. Math. Soc. **9** (1958), 382–389.

This paper investigates the linear ordering of the group of o-automorphisms of an abelian o-group. In particular, it is shown that the group of automorphisms can be linearly ordered if the rank of the o-group is well-ordered and the group of o-automorphisms of each component is a subgroup of the positive rationals.

H. A. Thurston (Vancouver, B.C.)

1341:

Wielandt, Helmut. Über den Normalisator der subnormalen Untergruppen. Math. Z. **69** (1958), 463–465.

The author calls a subgroup S of a group G subnormal if there exists a finite chain of subgroups, each normal in the preceding, which begins with G and ends with S. He denotes by $Ns(G)$ the intersection of the normalizers of all the subnormal subgroups of G. Then he forms an ascending series A of subgroups, beginning with the identity subgroup, by the requirement that the successor, V, of a term U of the series, must satisfy $U \subset V$, $V/U=Ns(G/U)$. He shows that if G satisfies the maximal condition for characteristic subgroups and the minimal condition for subnormal subgroups, then A reaches G in finitely many steps. The proof depends upon the fact that, for an arbitrary group G, $Ns(G)$ contains (a) every simple, non-abelian subnormal subgroup of G and (b) every minimal normal subgroup of G whose subnormal subgroups satisfy the minimal condition. He also shows that if S and E are subnormal subgroups of a group G and E is simple non-abelian, then S and E commute elementwise. This last result, and several others which we shall not quote, carry over to the infinite case the fundamental work of the author on subnormality in finite groups [H. Wielandt, Math. Z. **45** (1939), 209–244].

R. H. Bruck (Madison, Wis.)

1342:

Specht, Wilhelm. Beiträge zur Gruppentheorie. I. Lokalendliche Gruppen. Math. Nachr. **18** (1958), 39–56.

Consider a group G and the existence of a (transfinite) ascending sequence A of subgroups U of G, beginning with the identity subgroup and ending with G. Let U, V denote a typical element of A and its immediate successor, so that U is a proper subgroup of V. The group G is called (1) weakly metafinite if A exists with each U of finite index in V; (2) metafinite if A exists with each U normal of finite index in V; (3) J-metafinite if A exists with each U, V normal in G and each U of finite index in V. In each case, the existence of A implies the existence of an unrefinable A of the same type. In particular, J-metafinite groups are co-extensive with c-groups (in German, h-groups), namely groups possessing an ascending chief series with all of its quotient groups finite. Again, a group G is a c-group if and only if every homomorphic image $H\neq 1$ of G possesses a finite normal subgroup $F\neq 1$; thus, subgroups and homomorphic images of c-groups are c-groups.

For any group G, let $O(G)$ denote the union of all finite normal subgroups of G. By the theorem of Dietzmann (Dičman), $O(G)$ is the set of all elements of G having finite order and finite conjugate classes. A group G is a c-group if and only if A exists with $V/U=O(G/U)$ for every U. A group with $O(G)=G$ is called an oc-group (in German, an ok-group).

For any subgroup H of a group G, let H^* denote the centralizer of H in G. Then any ascending sequence A determines a descending sequence A^* (the successor of U^* is V^*) beginning with $E^*=G$ but ending with the centre $Z(G)=G^*$ of G. If G is oc and if A has property (3), then the factors U^*/V^* are finite; therefore, if G is oc, $G/Z(G)$ possesses a (transfinite) descending chief series with finite factor groups. Furthermore, if G is oc and if the element g of G is not in $Z(G)$, then there exists a subgroup of finite index in G which does not contain g.

If G is a c-group and A is an ascending normal series (each U normal in V) with finite factors V/U, the corresponding descending sequence A^* has the following property: to each U^* there corresponds a normal subgroup W^* of G such that $V^*\subset W^*\subset U^*$ and the quotient groups U^*/W^* and W^*/V^* are respectively finite and abelian. If, further, A has finite length k and $V/U=O(G/U)$ for each U, then the c-group G possesses a (transfinite) descending chief series whose factors, except for at most k abelian factors, are all finite.

The author shows in an appendix that the class of weakly metafinite groups properly contains the class of metafinite groups, that the class of metafinite groups properly contains the class of c-groups (J-metafinite groups) and that the class of c-groups properly contains the class of oc-groups.

{Reviewer's remark. Although the paper is self-contained, one must regret that the author gives no recognition to an extensive allied literature. All the references are to unnamed theorems in his book [W. Specht, *Gruppentheorie*, Springer, Berlin-Göttingen-Heidelberg, 1956; MR **18**, 189]. The reader may find it worthwhile to consult the bibliography in the English edition of Kurosh, *The theory of groups* [Vol. I, II, Chelsea, New York, 1955, 1956; MR **17**, 124; **18**, 188] as well as the more recent volumes of Mathematical Reviews.}

R. H. Bruck (Madison, Wis.)

1343:

★Lomont, J. S. Applications of finite groups. Academic Press, New York-London, 1959. xi+346 pp. $11.00.

This book ranges over a very wide field. In addition to the usual theory of representations and characters for finite groups the mathematical techniques discussed include reality conditions for representations and the little group technique for constructing irreducible representations. These general methods are applied to point groups, double point groups, space groups and their representations, the full rotation group, the symmetric groups, the full linear groups and the Lorentz group. Physical applications range through thermodynamics, molecular vibrations, energy bands in solids and the electronic structure of atoms and molecules.

It is perhaps inevitable that with such a coverage the treatment should be extremely condensed and at times superficial. The majority of the theorems used are quoted without proof. Some are illustrated by simple (sometimes trivial) examples, but at times the text degenerates into a mere list of results and formulae. It cannot, therefore, be recommended as an introduction to the subject. However, this superficiality is mitigated by two features. There are comprehensive bibliographies at the end of each chapter and a numbered set of references in the appendix. In addition the aspects of the theory which are treated in most cavalier fashion (the full rotation group, the symmetric groups, atomic and nuclear structure) are just those which are readily accessible in other texts.

Perhaps the most useful sections are those devoted to the theory of little groups and the irreducible representations of space groups. These representations have played an increasingly important role in recent studies of energy bands in solids and this account of the basic group-theoretical techniques required for their derivation will be welcomed by research workers in this field.

There are a number of misprints.

A. C. Hurley (Melbourne)

1344:

Brahana, H. R. Metabelian p-groups with five generators and orders p^{12} and p^{11}. Illinois J. Math. **2** (1958), 641–717.

In a previous paper [Amer. J. Math. **73** (1951), 539–555; MR **13**, 104], the author has produced a list of metabelian groups with elements of order p which are generated by five elements and are not direct products of abelian groups and metabelian groups with fewer generators. In the present paper he shows that that list is complete. *H. A. Thurston* (Vancouver, B.C.)

1345:

Hall, Marshall, Jr. Solution of the Burnside problem for exponent six. Illinois J. Math. **2** (1958), 764–786.

For positive integers n, k, let $B(n, k)=F/F^n$ where F is the free group of rank k and F^n is the subgroup generated by the nth powers of the elements of F. Then $B(n, k)$ is the free Burnside group of exponent n on k generators. The Burnside Problem for exponent n (not to be confused with several "restricted" Burnside Problems) is the question as to whether $B(n, k)$ is finite for every positive k. The only affirmative answers to date are as follows: $n=2$ (trivial); $n=3$ [Burnside, Quart. J. Pure Appl. Math. **33** (1902), 230–238; Levi and van der Waerden, Abh. Math. Sem. Univ. Hamburg, **9** (1933), 154–158]; $n=4$ [Sanov, Leningrad. Gos. Univ. Ped. Inst. Uč. Zap. Mat. Ser. **10** (1940), 166–170; MR **2**, 212]; and $n=6$ (the author in the present paper). Novikov has recently announced a negative answer for every $n \geq 72$ [Dokl. Akad. Nauk SSSR **127** (1959), 749–752].

The bare skeleton of the author's proof appears as follows: Let $G=B(6, k)$, $M=G^3$. Then $G/M=B(3, k)$, which is finite, so M is a group of exponent 6 generated by finitely many elements of order two. Hence M/M' is finite, and $H=M'$ has the following property: (*) H is a subgroup of a group M of exponent 6 and H is generated by finitely many elements of form $(ef)^2$ where the elements e, f of M have order two. The (dismayingly difficult) final step is to prove, assuming (*), that H has exponent 3 (and hence is finite).

The author reduces this final step to the case that H has four such generators (a relatively easy matter) and then to the corresponding three-generator case (a difficult reduction). There still remains a heroic piece of calculation before the end is in sight. The above-mentioned reductions of the final step are actually stated in more general terms, and the corresponding theorems are of interest in themselves.

Once he has proved that $B(6, k)$ is finite, the author needs only to refer to a paper of P. Hall and Higman [Proc. London Math. Soc. (3) **6** (1956), 1–42; MR **17**, 344] for the exact order of $B(6, k)$.

R. H. Bruck (Madison, Wis.)

1346:

Gol'berg, P. A. On a theorem of Wielandt. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 153–156. (Russian)

In Math. Z. **60** (1954), 407–408 [MR **16**, 331] Wielandt showed that if H is a nilpotent subgroup of a finite group, and has order h prime to its index, then any subgroup K of order dividing h is contained in a conjugate of H. It is known that certain apparently natural generalizations of this theorem are not valid; the author proves the following one. Let Π be a collection of natural prime numbers. Let H be a Π-Sylow subgroup of G, and let H have a finite number of conjugates and be the direct product of its Sylow subgroups. If M is a Π'-subgroup of G, $\Pi' \subset \Pi$, then M is contained in a conjugate of H. (By a Π-subgroup of G is meant a periodic group, the order of every element of which is divisible only by primes contained in Π. A Π-Sylow subgroup H of G is a Π-subgroup H such that for every $p \in \Pi$, a Sylow p-subgroup of H is a Sylow-p subgroup of G.)

J. L. Brenner (Menlo Park, Calif.)

1347:

Baer, Reinhold. Sylowturmgruppen. Math. Z. **69** (1958), 239–246.

In this review, "group" will mean "finite group". A group G is a Sylow-tower-group (ST-group) provided each factor-group $F\neq 1$ of G has a Sylow subgroup $S\neq 1$ such that S is normal in G. Equivalently, if $G\neq 1$, there exists at least one ordered set of primes $p_1, p_2, \cdots, p_k$ (not necessarily distinct) and an ascending "Sylow-series" of normal subgroups $S(i)$ such that $S(0)=1$, $S(k)=G$ and $S(i)/S(i-1)$ is a non-trivial p_i-Sylow-subgroup of $G/S(i-1)$ for $1\leq i\leq k$. Note that $S(i)$ is the characteristic subgroup consisting of all elements of G whose orders have only prime-factors contained among $p_1, \cdots, p_i$. The set $p_1, \cdots, p_k$ serves as a sort of "order-type" for the ST-group G.

Given the characteristic $c(G)$ of a group G—namely, the set of all prime divisors of the order $o(G)$ of G—what restrictions must be placed upon the "order-type" if G is to be ST? It is to this question (nontrivial since the direct product of ST-groups need not be ST) that the present paper is directed. A special rôle is played by the Hall subgroups. A subgroup H of a group G is a Hall subgroup provided the order of H is prime to its index in G.

The author introduces the following relation on $c(G)$: For distinct primes p, q in $c(G)$, p is stronger than q if there exists an inner automorphism of G which induces an automorphism of order p on some q-subgroup of G. It turns out that, for distinct p, q in $c(G)$, the number of p-Sylow-subgroups of G is prime to q precisely when there exists a Hall ST-subgroup H of G of characteristic p, q such that, in $c(H)$, p is not stronger than q.

The "stronger than" relation is (by definition) not reflexive and (by an example due to Philip Hall) not necessarily transitive. Hence the author defines the valuation of $c(G)$ to be the transitive extension of the "stronger than" relation.

Next the author considers an arbitrary set $\mathfrak{R}$ of primes and defines a group G to be an $\mathfrak{R}$-group if $c(G)$ is in $\mathfrak{R}$. (An $\mathfrak{R}$-element is similarly defined.) A semi-ordering σ of $\mathfrak{R}$ is a binary relation on $\mathfrak{R}$ which is transitive but anti-circular ($p\sigma p$ is always false.) A σ-tail of $\mathfrak{R}$ is a subset $\mathfrak{S}$ of $\mathfrak{R}$ which contains, with p, each x in $\mathfrak{R}$ such that $x\sigma p$. The $\mathfrak{R}$-group G is σ-dispersed if, for each σ-tail $\mathfrak{S}$, the set $G_{\mathfrak{S}}$ of all $\mathfrak{S}$-elements is a (characteristic) subgroup of G. It turns out that G is an ST-group if and only if there exists a semi-ordering σ of $c(G)$ such that G is σ-dispersed. Theorem 1: G is an ST-group if and only if the valuation of $c(G)$ is a semi-ordering.

Let $\mathfrak{R}$ be a prime set and G any group (not necessarily an $\mathfrak{R}$-group). Then the valuation of $c(G)$ induces a relation $\sigma_{\mathfrak{R},G}$ on $\mathfrak{R}$ (more precisely, on $\mathfrak{R}\cap c(G)$.) The group G is $\mathfrak{R}$-regular provided (a) $\sigma_{\mathfrak{R},G}$ is a semi-ordering of $\mathfrak{R}$, and (b) if U is a subgroup of G and if p, q is a pair of primes in $\mathfrak{R}\cap c(U)$, then U possesses a Hall subgroup of characteristic p, q. Theorem 2: The group G is $\mathfrak{R}$-regular if and only if the following conditions are satisfied: (S) Each subgroup U of G possesses a Hall ST-subgroup of characteristic $\mathfrak{R}\cap c(U)$; (K) If H is a Hall $\mathfrak{R}$-subgroup of G and if V is a subgroup of G such that $o(V)$ divides $o(H)$, then V is conjugate to a subgroup of H.

R. H. Bruck (Madison, Wis.)

1348:

Blackburn, N. On prime-power groups with two generators. Proc. Cambridge Philos. Soc. **54** (1958), 327–337.

Throughout this paper G is a finite p-group; $d(G)$ is the minimal number of generators of G; G' and $\phi(G)$ are the commutator subgroup and Frattini subgroup, respectively, of G; and, for each positive integer t, G_t is that term of the lower central series of G which is generated by the extended commutators of length t with arguments ranging over G. The main theorems are as follows. Theorem 2.1: If $d(G)=2$ and if G/G' has type (m, n), then $d(G')\leq(p^m-1)(p^n-1)$ and $G_k\subset\phi(G')$ for $k=p^m+p^n-1$. These results are best possible. Theorem 2.3: A p-group G is metacyclic if and only if $G/\phi(G')G_3$ is metacyclic. Theorem 2.6: (i) For p odd, a p-group G is metacyclic if and only if G^p (the subgroup generated by the pth powers) has index at most p^2 in G; (ii) (Huppert) for p odd, a p-group which can be expressed as a product of two cyclic groups is metacyclic; (iii) for all p, a p-group which has not more than $p+1$ subgroups of index p^2 is metacyclic. Theorem 2.7: (Itô and Ôhara) If G is a 2-group which can be expressed as a product of two cyclic groups, and if the invariants of G/G' are both greater than 1, then G is metacyclic.

In the above an abelian p-group has type (m, n) if it is the direct product of two cyclic groups of orders p^m and p^n. And a group G is metacyclic if it has a normal subgroup N such that $\cancel{G}$ and G/N are cyclic. / N

In the first part of the paper the author considers a p-group G such that G' is elementary abelian and G/G' has type (m, n), $m\leq n$. Let a, b be generators of G having orders p^m, p^n, respectively, modulo G'. By imposing upon the good nature of commutators in metabelian groups, the author obtains the following sharp bounds for $d_t=d(G_t/G_{t+1})$. (i) If $2\leq t\leq p^m+1$, then $d_t\leq t-1$; moreover, if $d_t\geq t-2$ for some such t, then $d_u=u-1$ for $2\leq u\leq t-1$, and a^{p^m}, b^{p^n} lie in G_t. (ii) If $p^m+1\leq t\leq p^n+1$, then $d_t\leq p^m-1$; moreover, if $d_t=p^m-1$ for some such t, then $d_u=p^m-1$ for $p^m\leq u\leq t-1$, b^{p^n} lies in G_t, and, for $r\geq 1$, the extended commutators with p^m arguments a and r arguments b lie in G_{s+r-1}, where $s=p^m$. (iii) For $p^n+1\leq t\leq p^n+p^m-2$, $d_t\leq p^m+p^n-t-1$; and $G_k=1$ where $k=p^m+p^n-1$. These computational labours constitute the main "dirty work" of the paper, and the author is able to go on from them with relatively clean hands.

R. H. Bruck (Madison, Wis.)

1349:

Blackburn, N. On a special class of p-groups. Acta Math. **100** (1958), 45–92.

Let $\gamma_2(G)$ denote the commutator subgroup of the group G, and let $G, \gamma_2(G), \gamma_3(G), \cdots$, be the terms of the lower central series of G. The main theme of this paper is the study of a certain judiciously selected class of finite p-groups, namely the class $\mathrm{CF}(m, n, p)$ of all groups of order p^n, nilpotency class $m-1$ such that the factor groups $\gamma_2(G)/\gamma_3(G), \cdots, \gamma_{m-1}(G)/\gamma_m(G)$ all have order p.—The methods have a wider application which we shall not indicate here.—The subclass $\mathrm{ECF}(m, n, p)$ consists of all G in $\mathrm{CF}(m, n, p)$ such that the abelian group $G/\gamma_2(G)$ has exponent p. It will be observed that if $\mathrm{CF}(m, n, p)$ contains a non-abelian group G, then $3\leq m\leq n$ and $G/\gamma_2(G)$ has order p^{n-m+2}. In particular, $\mathrm{CF}(n, n, p)$ and $\mathrm{ECF}(n, n, p)$ coincide with the class of groups of order p^n and maximal

class $(n-1)$ consistent with that order. Such "groups of maximal class" have a rigid "central" structure—their upper and lower central series coincide—and they also have a decisive rôle to play in the study of $ECF(n, m, p)$.

The methods of the paper do not apply to the metabelian groups $CF(3, n, p)$—moreover, $ECF(3, n, p)$ has been dissected by Schreier [Abh. Math. Sem. Univ. Hamburg **4** (1926), 321–346]—so the author assumes $3 < m \leq n$.

The author introduces two new characteristic subgroups. $\gamma_1(G)$ is not G, as one would expect; instead, $\gamma_1(G)/\gamma_4(G)$ is the centralizer of $\gamma_2(G)/\gamma_4(G)$ in $G/\gamma_4(G)$. And $\eta(G)$ is defined so that $\eta(G)/\gamma_3(G)$ is the centre of $G/\gamma_3(G)$. Then, for G in $CF(m, n, p)$, $\eta(G)$ is the last term but one (the last being G) of the upper central series of G. Moreover, the sequence G, $\gamma_1(G)$, $\eta(G)$, $\gamma_2(G)$, $\gamma_3(G)$, $\cdots$, $\gamma_m(G)$ is descending and has all its factors of order p except that $\eta(G)/\gamma_2(G)$ has order p^{n-m+1}.

The group G in $CF(m, n, p)$ is said to have degree of commutativity k if, for all i, $j \geq 1$, the commutator subgroup $[\gamma_i(G), \gamma_j(G)]$ is in $\gamma_{i+j+k}(G)$. Every G has degree (of commutativity) 0, and the interesting question is as to when G also has positive degree. It turns out that, in either of the following cases, every G in $CF(m, n, p)$ has positive degree: (i) m odd, $5 \leq m \leq 2p+1$; (ii) m even, $6 \leq m \leq 2p+2$. In addition, every G in the subclass $ECF(m, n, p)$ has positive degree when $m > p+1$. These results are best possible. However, if G is in $ECF(m, n, p)$, if $m \geq p+1$, and if $\gamma_2(G)$ is abelian, then G has degree $m-p$ (and there are analogous results when γ_2 is replaced by some later term). In addition, sharper results are given for $p = 2, 3, 5$.

Most of the proofs involve calculation with commutators—and, for this reason, as the author points out, a number of the theorems have generalizations to wider classes of (finite or infinite) groups. One important tool is a formula expressing $(xy)^p$, modulo a certain subgroup H, in the explicit form that would result if y lay in an abelian normal subgroup. The author provides an excellent estimate for H by a skillful use of the Hall collection process [Philip Hall, Proc. London Math. Soc. (2) **36** (1933), 29–35].

The author concludes the paper by determining the number of types of groups G of order p^n and (maximal) class $n-1$ under certain conditions. The results are complete for $p=3$; for $p>3$, $n=6$; and for G metabelian with $[\gamma_1(G), \gamma_2(G)]$ in $\gamma_{n-2}(G)$.

In connection with the results of the last paragraph and, more generally, with the groups of maximal class, the author expresses his indebtedness to A. Wiman [Acta Math. **88** (1952), 317–346; MR **14**, 722] for certain ideas. The two papers do not agree, and the author comments that Parts II and III of Wiman's paper contain statements "which seem to us to be in general untrue". *R. H. Bruck* (Madison, Wis.)

1350:

Wohlfahrt, Klaus. Über Dedekindsche Summen und Untergruppen der Modulgruppe. Abh. Math. Sem. Univ. Hamburg **23** (1959), 5–10.

In this paper the author constructs subgroups of the modular group of finite index which are not congruence groups.

Let $\Gamma_0(n)$ be the subgroup of the full modular group Γ for which $\gamma \equiv 0 \pmod{n}$. With the help of Dedekind's multiplicator system for the modular group, the author defines a character χ on $\Gamma_0(n)$. If $L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then using the reciprocity law for Dedekind Sums, he deduces

$$\chi(L) = \exp \frac{\pi i r}{6} \cdot \frac{(n-1)(\beta+\gamma n^{-1}) + 2\{\gamma n^{-1}, \delta\} - 2\{\gamma, \delta\}}{\delta},$$

where $\{j, h\} = 6|h|s(j, h)$, and r is arbitrary. He then proves the following theorem: The kernel Δ of χ with n a prime p, $r = 12/q$ (q a prime), $p^2 \not\equiv 0,1 \pmod{q}$ is not a congruence subgroup of the modular group.

R. Ayoub (State College, Pa.)

1351:

Tamura, Takayuki. The theory of construction of finite semigroups. III. Finite unipotent semigroups. Osaka Math. J. **10** (1958), 191–204.

[For parts I and II, see same J. **8** (1956), 243–261; **9** (1957), 1–42, 242; MR **18**, 717; **19**, 940.]

In part II, the author defined unipotent semigroup and z-semigroup. In the present paper he continues the investigation of their structure, and finds a construction (in terms of z-semigroups) of all unipotent semigroups which contain non-trivial groups as sub-semigroups.

{The editors received the following correction from the author: On p. 198, l. 10, for "$x, y \in I_i$" read $x \overset{i}{\sim} y \in I_i$".}

H. A. Thurston (Vancouver, B.C.)

1352:

Tamura, Takayuki. Characterization of certain additive semigroups by distributive multiplications. J. Gakugei Tokushima Univ. Math. **9** (1958), 21–24.

The semigroup $S(+)$ is right [left] singular if $x+y = y$ [$x+y=x$] for all x, y in S. $S(+)$ is a zero semigroup if there is an element 0 in S such that $x+y=0$ for all x, y in S. The author characterizes (i) left or right singular semigroups S and (ii) zero semigroups S in terms of the possible binary multiplications distributive over + which can be defined on S. *G. B. Preston* (Shrivenham)

1353:

Chaudhuri, Niranjan Prasad. Sur les complexes unitaires dans un demi-groupe. C. R. Acad. Sci. Paris **248** (1959), 1750–1752.

The author investigates conditions under which unitary subsets of semigroups are also subsemigroups. A typical theorem is the following. In a semigroup which is a union of disjoint groups and in which the left cancellation law holds, every left unitary subset is a subsemigroup.

G. B. Preston (Shrivenham)

1354:

Kimura, Naoki. The structure of idempotent semigroups. I. Pacific J. Math. **8** (1958), 257–275.

A band is an idempotent semigroup (one satisfying the identity $a^2 = a$). In the study of bands, special classes of bands characterized by an identity play a special rôle. An important type is the semi-lattice ($ab = ba$). For each of the other classes (as the author shows towards the end of the paper, where he obtains all suitable identities) there are infinitely many choices for a single identity characterizing the class in the class of all bands. We merely mention the class and the simplest defining identity: rectangular bands ($aba = a$), left singular bands ($ab = a$), regular bands

($abaca=abca$), left regular bands ($aba=ab$) and the analogously defined right singular and right regular bands.

Left singular bands are both rectangular and left regular; similarly with "left" replaced by "right". Left regular, right regular and rectangular bands are all regular. Each rectangular band is representable (essentially uniquely) as a direct product of a left singular and a right singular band. These are the more elementary results.

At the end of the paper the author defines the free band of each type on n generators and calculates its order.

As an interesting special remark: A semigroup S satisfying the identity $abc=ac$ need not be a band, but SS is a regular band. Moreover, the identity characterizes regular bands among bands.

Before giving the main theorem of the paper, we will describe briefly a general situation which is discussed more fully in chapter 2, § 8 of the reviewer's book [R. H. Bruck, *A survey of binary systems*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958; MR **20** #76]. Let S be a semigroup without zero such that x is in Sx^2S for each x in S (for example, let S be a band). For each x in S, let F_x be the set of all y in S such that $SyS=SxS$. Then each F_x is a simple sub-semigroup of S and the mapping $x\to F_x$ is the natural homomorphism of S upon a semi-lattice. (Here the names of David McLean, Olaf Andersen, Robert Croisot, A. H. Clifford and J. A. Green should certainly be mentioned, and possibly several others.)

When S is a band, each F_x is a rectangular band (McLean). The author points out that each F_x is a maximal rectangular sub-band. Moreover, if the band S belongs to one of the classes described above, so must each F_x.

The author's main contribution is the concept of the spined product. Let A, B be bands having natural homomorphisms p, q, respectively, upon the same semi-lattice. Then the spined product S of A and B is that sub-semigroup S of the direct product $A\times B$ consisting of all ordered pairs (a, b), a in A, b in B, such that $p(a)=q(b)$. The author's main theorem (which, according to a footnote, seems to have been discovered independently by Miyuki Yamada) may now be stated as follows: A band is regular if and only if it is the spined product of a left regular band and a right regular band. *R. H. Bruck* (Madison, Wis.)

1355:

Klein-Barmen, Fritz. Ordoid, Halbverband und ordoide Semigruppe. Math. Ann. **135** (1958), 142–159.

The main virtue of this paper—its clear and leisurely discussion of various axiom systems for classes of commutative semigroups—cannot be communicated in a brief review. We shall be content to introduce enough definitions for the statement of Theorem 23.1. Let M be a commutative semigroup with respect to an operation $\cap$. (The author uses the term "commutative associative".) Further, let M be an ordoid with respect to an operation $\cup$; that is, a commutative semigroup satisfying: IV. For each x in M there exists at least one y in M such that $x=x\cup y$. V. If $x=y\cup y'$ and $y=x\cup x'$, then $x=y$. (IV and V allow the introduction of a reflexive and transitive "inequality relation" $\geqq$ as follows: $x\geqq y$ provided $x=y\cup y'$ for at least one y'. Then $\supset$ and the "reversed inequalities" $\leqq$, $\subset$ are defined in the obvious manner.) And finally, assume the distributive law $x\cap(y\cup z)=(x\cup y)\cap(x\cup z)$ for all x, y, z in M. Conclusion: $x\supset x\cap(x\cup y\cup z)$ implies $x\cup(y\cap z)\geqq(x\cup y)\cap(x\cup z)$; and the result remains true when both inequalities are reversed or both are replaced by equality.

The paper has a bibliography of about 30 items.

R. H. Bruck (Madison, Wis.)

1356:

Klein-Barmen, Fritz. Verallgemeinerung des Verbandsbegriffs durch Abschwächung des Axioms der Idempotenz. Math. Z. **70** (1958), 38–51.

An expository article, covering much the same ground as #1355 above. *R. H. Bruck* (Madison, Wis.)

1357:

Felscher, W.; und Klein-Barmen, F. Zur Axiomatik der kommutativen Halbverbände. Arch. Math. **10** (1959), 7.

On a commutative semigroup the law (I) $xx=x$ is equivalent to the pair of laws (U) $x(yz)=(xy)(xz)$ and (V) for each x there exists y with $xy=x$. However, (V) implies neither (U) nor (I); and (U) implies neither (V) nor (I). The law (U) was previously studied by Felscher [Arch. Math. 8 (1957), 171–174; MR **19**, 1154].

R. H. Bruck (Madison, Wis.)

1358a:

Kimura, Naoki. On some existence theorems on multiplicative systems. I. Greatest quotient. Proc. Japan Acad. **34** (1958), 305–309.

1358b:

Kimura, Naoki. On some existence theorems on multiplicative systems. II. Maximal subsystems. Proc. Japan Acad. **34** (1958), 310–314.

These two papers are essentially abstracts (with definitions and theorems but no proofs) of parts I and II of a paper entitled "On multiplicative systems", to be published elsewhere. *R. H. Bruck* (Madison, Wis.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 1586, 1587, 1588, 1589, 1628.

1359:

Ivanovskiĭ, L. N. On a conjecture of P. S. Alexandrov. Dokl. Akad. Nauk SSSR **123** (1958), 785–786. (Russian)

Let G be a compact topological group. According to Pontrjagin [*Nepreryvnye gruppy*, 2nd ed. Gosudarstv. Izdat.-Tehn.-Teor. Lit., Moscow, 1954; MR **17**, 171] there is a "Lie series" S_θ which is an inverse mapping system $\{G_\alpha, \pi_\beta{}^\alpha, \beta<\alpha<\theta\}$ (θ is the first ordinal of power equal to the weight m of G), where G_1 is a Lie group, and the mappings $\pi_\alpha{}^{\alpha+1}$ are onto, with kernels also Lie groups. The limit group is G. Using this representation, a mapping of K^m onto G is constructed, where K is the Cantor set.

R. Arens (Los Angeles, Calif.)

1360:

Irie, Shoji. On Haar measure. Sûgaku **9** (1957/58), 99–109. (Japanese)

An introduction to a lecture by J. Dieudonné on Haar measure of a locally compact group at University of Paris.

H. Yamabe (Osaka)

1361:

Boclé, Jean. Théorèmes de dérivation globale dans les groupes topologiques localement compacts. C. R. Acad. Sci. Paris **248** (1959), 2063–2065.

Let φ be a Radon measure on a locally compact group, and let λ be left-invariant Haar measure. The author shows that as B converges to the identity over symmetric neighborhoods of the identity, the function $\varphi(Bt)/\lambda(Bt)$ converges in measure to $(d\varphi/d\lambda)(t)$. This generalizes problem (5) on page 268 of Paul R. Halmos, *Measure theory* [Van Nostrand, New York, 1950; MR **11**, 504].

W. F. Stinespring (Princeton, N.J.)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

1362:

Hofmann, Karl Heinrich. Topologische Loops. Math. Z. **70** (1958), 13–37.

Dies ist die grundlegende einer Reihe von Arbeiten des Verf., in denen er topologische nichtassoziative algebraische Strukturen mit einer und zwei Verknüpfungen untersucht, zwar losgelöst von geometrischen Betrachtungen, aber doch im Hinblick auf die Anwendung auf topologische Inzidenz-Strukturen. Hier werden die mehr formalen Begriffe und Sätze aus der Theorie der topologischen Gruppen [wie sie in großen Zügen etwa den Inhalt der §§ 17–23 und 50–51 der 2. Aufl. von L. S. Pontrjagin, *Nepreryvnye gruppy*, Gosudarstv. Izdat. Tehn. Teor. Lit., Moscow, 1954; deutsch, Teubner, Leipzig, 1957/58; englisch, Princeton Univ. Press, 1958; MR **17**, 171; **19**, 152; **20** #3925; **19**, 867; bilden] mit geeignet abgeänderten Beweisverfahren, soweit es möglich ist, auf topologische Loops übertragen, und die speziellen Schwierigkeiten und Abweichungen diskutiert, die durch das Fehlen des Assoziativgesetzes entstehen.—Nach den grundlegenden Definitionen von topologischen Loops und ihren Abbildungen und Homomorphismen werden unter anderen folgende Eigenschaften bewiesen. Eine offene Unterloop einer topologischen Loop ist auch abgeschlossen (dies gilt, obwohl eine Unterloop keine Zerlegung in Nebenklassen liefert). Ist A eine abgeschlossene und K eine kompakte Teilmenge einer topologischen Loop, so ist das Komplexprodukt AK abgeschlossen; falls $A\cap K=\emptyset$ ist, gibt es eine Umgebung U des Neutralelementes mit $AU\cap KU=\emptyset$. Wie zuerst A.I. Mal'cev [Mat. Sb. (N.S.) **35 (77)** (1954), 3–20; MR **16**, 440] bemerkt hat, ist jede T_1-Loop sogar regulär. Ob sie wie eine Gruppe auch vollständig regulär ist, bleibt eine offene Frage. Jedenfalls bilden im Gegensatz zu Gruppen die Mengen der Paare (x, y) mit $y\in Ux$ im allgemeinen keine Basis für eine mit der Topologie verträgliche uniforme Struktur, wenn U den Umgebungsfilter des Neutralelementes durchläuft. Ein Beispiel zeigt, daß das nicht einmal für lokal kompakte Loops der Fall ist. Für die eindeutig bestimmte uniforme Struktur einer kompakten topologischen Loop besteht jedoch der von den Gruppen her gewohnte Zusammenhang mit den Umgebungen des Neutralelementes.—Betrachtet man in einer zusammenhängenden, lokal kompakten topologischen Loop die Zusammenhangs-Komponenten der Punkte in allen ihren Umgebungen als neue Umgebungen bezüglich einer feineren Topologie, so wird die Loop mit der neuen Topologie zu einer nicht diskreten, lokal kompakten und lokal zusammenhängenden topologischen Loop, die überdies kurvenweise zusammenhängend ist, wenn die Loop in der ursprünglichen Topologie kurvenweise zusammenhängend war.—Schließlich wird ohne Beweise die Theorie der Überlagerungsräume nach Cl. Chevalley [*Theory of Lie groups*, Princeton, Univ. Press, 1946; MR **7**, 412] skizziert und für jede topologische Loop, deren zugrunde liegender topologischer Raum einen einfach zusammenhängenden Überlagerungsraum besitzt, die Existenz einer bis auf Isomorphie eindeutig bestimmten universellen Überlagerungs-Loop bewiesen. Jede zusammenhängende, lokal einfach zusammenhängende topologische Loop hat eine kommutative Fundamentalgruppe. {Im Literatur-Verzeichnis sind versehentlich die Namen Hewitt und Novak vertauscht.}

H. Salzmann (Frankfurt)

1363:

Hofmann, Karl Heinrich. Topologische Loops mit schwachen Assoziativitätsforderungen. Math. Z. **70** (1958), 125–155.

In dieser Arbeit, die an die vorstehend besprochene Arbeit anknüpft, wird die Frage untersucht, wie weit in topologischen, insbesondere lokal kompakten Loops bei geeigneten topologischen Voraussetzungen aus gewissen Abschwächungen des Assoziativ-Gesetzes weitere Teilaussagen des vollen Assoziativ-Gesetzes folgen.—Ein Element einer Loop wird monassoziativ genannt, wenn es in einer Unterhalbgruppe der Loop liegt, potenzassoziativ, wenn es in einer Untergruppe liegt; ein Elementepaar heißt diassoziativ, wenn es in der Loop eine Halbgruppe erzeugt, und alternativ, wenn es ein und derselben Untergruppe der Loop angehört. Eine Loop heißt monassoziativ usw., wenn jedes ihrer Elemente bzw. Elementepaare die entsprechende Eigenschaft hat, lokal monassoziativ usw., wenn das wenigstens für alle Elemente bzw. Elementepaare einer Umgebung des Neutralelementes gilt. Als geringste Abschwächung des Alternativ-Gesetzes wird die Moufang-Identität betrachtet: Eine Loop heißt Moufangsch, wenn in ihr die Identität $x(y(zy))=((xy)z)y$ gilt. Eine topologische Loop, deren Neutralelement Häufungspunkt monassoziativer Elemente ist, bezeichnet Verf. als eudoxisch, wenn das Neutralelement eine Umgebung besitzt, die keine nicht triviale Unterhalbgruppe der Loop ganz enthält. Er beweist unter anderem: Die Menge der monassoziativen Elemente einer topologischen Loop ist eine abgeschlossene Teilmenge; Entsprechendes gilt für die Begriffe potenzassoziativ, diassoziativ und alternativ. Eine zusammenhängende, eudoxische, kommutative Moufangloop ist eine Gruppe. Ein kompakter Normalteiler (=Kern eines Homomorphismus) einer lokal kompakten, zusammenhängenden kommutativen Moufangloop liegt im Kern (=Menge der Elemente der Loop, die mit allen Elementepaaren assoziieren).—Einen großen Platz in der Arbeit nimmt die Frage nach der Existenz und Eindeutigkeit von Einparametergruppen (=nicht trivialen homomorphen Bildern der additiven Gruppe der reellen Zahlen) ein. Ein zusammenfassendes Ergebnis hierüber (dessen Teilaussagen unter wesentlich schwächeren Voraussetzungen bewiesen werden) lautet: In einer lokal euklidischen, eudoxischen, lokal torsionsfreien, lokal alternativen, kommutativen Loop gibt es eine Umgebung des Neutralelementes, in der jeder Punkt auf genau einer Einparametergruppe liegt. Dabei soll eine eudoxische Loop lokal

torsionsfrei heißen, wenn es eine kompakte Umgebung U des Neutralelementes gibt, so daß UU nur aus potenzassoziativen Elementen besteht, die sämtlich unendliche Gruppen erzeugen. Die weiteren Strukturuntersuchungen eudoxischer, kommutativer Alternativloops beruhen nun darauf, daß sie mit Hilfe der Einparametergruppen lokal als "Fastvektorräume" über dem Körper der reellen Zahlen aufgefaßt werden können. Ein Fastvektorraum ist dabei eine potenzassoziative Loop mit einem Körper als Endomorphismenbereich, so daß für alle Elemente x der Loop und alle r, s aus dem Körper $(r+s)\cdot x=r\cdot x+s\cdot x$, $(rs)\cdot x=r\cdot(s\cdot x)$ und $1\cdot x=x$ gilt. So werden unter anderen folgende Strukturaussagen gewonnen: In einer lokal euklidischen, eudoxischen, lokal torsionsfreien, kommutativen, lokal alternativen Loop bildet die Vereinigung aller Einparametergruppen einen charakteristischen, zu einem reellen Fastvektorraum isomorphen Normalteiler mit diskreter Faktorloop. In einer kompakten, eudoxischen, kommutativen Alternativloop bildet die Vereinigung aller Einparametergruppen die Zusammenhangskomponente des Neutralelementes und diese ist eine endlichdimensionale Torusgruppe, also insbesondere assoziativ.—Eine zusammenhängende Loop mit einer zu einem reellen Intervall homöomorphen Umgebung des Neutralelementes ist assoziativ, wenn das Neutralelement Häufungspunkt potenzassoziativer Elemente ist oder wenn die Loop kompakt und das Neutralelement Häufungspunkt monassoziativer Elemente ist. Dagegen gibt es eine zum Raum der reellen Zahlen homöomorphe monassoziative Loop, die nicht potenzassoziativ ist.—Zum Schluß gibt Verf. ein Beispiel einer zum dreidimensionalen euklidischen Raum homöomorphen kommutativen echten Alternativloop, deren Verknüpfungen in keinem Koordinatensystem analytisch sein können, da nach A. I. Mal'cev [Mat. Sb. (N.S.) **36**: (78) (1954), 569–576; MR **16**, 997] jede analytische kommutative Alternativloop eine Gruppe ist.

H. Salzmann (Frankfurt)

1364:

Hofmann, Karl Heinrich. Topologische Doppelloops. Math. Z. **70** (1958), 213–230.

In dieser Arbeit werden die Untersuchungen zweier früherer Arbeiten des Verf. [#1362, #1363] ausgedehnt auf ringartige topologische Strukturen K mit zwei Verknüpfungen, die als Addition und Multiplikation geschrieben werden. Mit der Addition soll K eine Loop mit Neutralelement 0 bilden und es soll stets $0x=x0=0$ sein. Bildet $K^*=K-\{0\}$ mit der Multiplikation eine Loop, so wird die Struktur als Doppelloop bezeichnet. Verf. beweist: Eine topologische Doppelloop ist entweder zusammenhängend oder nirgends zusammenhängend. Eine lokal kompakte, zusammenhängende topologische Doppelloop ist lokal zusammenhängend. Eine zusammenhängende, lokal einfach zusammenhängende topologische Doppelloop ist einfach zusammenhängend. Eine topologische, kurvenweise zusammenhängende Doppelloop ist einfach zusammenhängend im Sinne von R. S. Novosad [Trans. Amer. Math. Soc. **79** (1955), 216–228; MR **17**, 393]. Die multiplikative Loop einer einseitig distributiven, lokal zu einem reellen Intervall homöomorphen Doppelloop, deren sämtliche Elemente additive Halbgruppen erzeugen (=monassoziativ sind), ist isomorph zur multiplikativen Gruppe der reellen Zahlen. Erzeugen die Elemente sogar additive Gruppen (d.h. ist die additive Loop potenzassoziativ), so ist die Doppelloop isomorph zum Körper der reellen Zahlen; die Zusatzvoraussetzung ist nicht entbehrlich. Eine lokal kompakte, zusammenhängende, beiderseits distributive, nullteilerfreie ringartige Struktur K, die bezüglich der Addition eine eudoxische (d.h. eine gewisse Umgebung von 0 enthält keine volle nicht triviale Halbgruppe) Moufangloop bildet, und die entweder lokal euklidisch oder eine Doppelloop ist, ist eine nicht notwendig assoziative Algebra endlichen Ranges über dem Körper der reellen Zahlen. Ist K^* sogar eine beiderseits kürzbare Loop, so ist K isomorph zum Körper der reellen oder komplexen Zahlen oder Quaternionen oder zur reellen Cayley-Algebra. Eine lokal euklidische, zweidimensionale, beiderseits distributive Doppelloop mit potenzassoziativer eudoxischer Addition und Multiplikation ist ein Neokörper, dessen multiplikative Gruppe zur multiplikativen Gruppe der komplexen Zahlen isomorph ist. Ist die additive Loop sogar alternativ, so ist die Doppelloop isomorph zum Körper der komplexen Zahlen. Ein Beispiel zeigt die Notwendigkeit der letzten Zusatzvoraussetzung.

H. Salzmann (Frankfurt)

1365:

Anderson, Lee W. On the distributivity and simple connectivity of plane topological lattices. Trans. Amer. Math. Soc. **91** (1959), 102–112.

A. D. Wallace conjectured that every compact, connected, topological lattice is distributive. It turns out that things are more complicated. D. E. Edmondson [Proc. Amer. Math. Soc. **7** (1956), 1157–1158; MR **18**, 461] showed that there exist subsets of Euclidean three-space which, with the appropriate lattice operations, form nonmodular, compact, connected, topological lattices. The author shows that such subsets do not exist in Euclidean two-space; that is, if a locally compact subset of a Euclidean two-space admits continuous lattice operations, then the lattice is distributive. This paper also contains some examples of topological lattices.

J. Hartmanis (Schenectady, N.Y.)

FUNCTIONS OF REAL VARIABLES

See also 1687.

1366:

Mioduszewski, J. Solution générale d'un problème de Sikorski. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 169–173.

R. Sikorski [Colloq. Math. **4** (1957), 240] proposed the following problem: If the real continuous functions f_1, f_2 are defined on [0, 1] and have the properties (1) $f_k(0)=0$, $0\leq f_k(x)\leq 1$, $f_k(1)=1$ for $k=1, 2$, and (2) neither f_1 nor f_2 is constant in any interval, is then every component of the set $A_2=\{(x,y)\,|\,f_1(x)=f_2(y)\}$ locally connected? This problem was affirmatively solved by J. S. Lipiński [Bull. Acad. Polon. Sci. Cl. III **5** (1957), 1019–1021; MR **20** #1730] and also the case $n\geq 2$ was discussed (under supplementary assumptions) by R. Sikorski and K. Zarankiewicz [Fund. Math. **41** (1954), 339–344; MR **17**, 288; see also Sikorski, ibid. 345–350; MR **17**, 289]. The author now generalizes the problem and its answer by dropping the assumption (1) and by determining the

precise structure (for $n \geq 2$) of the components of the set

$$A_n = \{(x_1, x_2, \cdots, x_n) \,|\, f_1(x_1) = f_2(x_2) = \cdots = f_n(x_n)\}.$$

The author calls a set E of order $\leq n$ in the strong sense at the point p if there exists a strictly monotone decreasing sequence of neighborhoods G of p such that $p = \bigcap \bar{G}$ and the frontier of each $\bar{G}$ intersects E in at most n points. Then the author obtains his main result: If the continuous functions $f_1, f_2, \cdots, f_n$ defined on $[0, 1]$ are not constant in any interval, then the above set A_n is of order $\leq 2^n$ in the strong sense at each of its points. He also shows that this result cannot be improved.

A. Rosenthal (Lafayette, Ind.)

1367:

Froda, Alexandre. Aspects abstraits d'une propriété des fonctions réelles sur un support sans structure. C. R. Acad. Sci. Paris **246** (1958), 2994–2996.

The author first proves the following two theorems: Every real function $f(P)$ defined in a domain Δ_n of an n-dimensional space has the following two properties. (1) There exists an at most countable set $D \subset \Delta_n$ such that, for every $\varepsilon > 0$ and to every point $P \notin D$, there corresponds at least one $Q \neq P$, $Q \in \Delta_n$, such that $|f(P) - f(Q)| < \varepsilon$. (2) There exists an at most countable set $D^* \subset \Delta_n$ such that, for every $P \notin D^*$, $f(P)$ is one of the limits of $f(Q)$ when Q tends to P.

These two theorems are then generalized as follows. Let $\mathfrak{A}$ be a space without any structure, $\mathfrak{R}$ a topological space satisfying the axiom (T_2) and the first axiom of denumerability. Let $\aleph_\lambda$ be the infinite cardinal number of a base $\mathfrak{B}$ of $\mathfrak{R}$ and call $\aleph_\rho = \inf \aleph_\lambda$ the "cardinal character" of $\mathfrak{R}$. (3) If $f(P)$ has $\mathfrak{A}$ as support and takes its values in a metric space $\mathfrak{M}$ of cardinal character $\aleph_\mu$, then there exists a set $D \subset \mathfrak{A}$ with card $D \leq \aleph_\mu$ such that $f(P)$ has the property: If $P \in \mathfrak{A} - D$ and $\varepsilon > 0$ is arbitrary, then there corresponds to P at least one $Q \in \mathfrak{A}$, $Q \neq P$, such that $d[f(P), f(Q)] < \varepsilon$. There follows a generalization of a well-known theorem of H. Blumberg [Trans. Amer. Math. Soc. **24** (1922), 113–128], namely: (4) If $f(P)$ is defined on a metric space $\mathfrak{E}$ and takes its values in a metric space $\mathfrak{M}$, then there exists a set $E \subset \mathfrak{E}$ dense in $\mathfrak{E}$ such that the restriction of $f(P)$ to E is continuous.

(5) Let $f(P)$ be given on the above-defined topological space $\mathfrak{R}$ whose cardinal character is $\aleph_\rho$. If $f(P)$ takes its values in a metric space $\mathfrak{M}$ with cardinal character $\aleph_\mu$, then there exists a set $D^* \subset \mathfrak{R}$ with card $D^* \leq \sup\{\aleph_\rho, \aleph_\mu\}$ such that $f(P)$ is one of the limits of $f(Q)$ at the point $P \in \mathfrak{R} - D^*$ if Q tends to P in $\mathfrak{R}$.

A. Rosenthal (Lafayette, Ind.)

1368:

Lepina, È. I. A generalized Green formula. Latvijas Valsts Univ. Zinātn. Raksti **20** (1958), no. 3, 125–135. (Russian. Latvian summary)

The author considers formulae (in particular Green's formula) in the form

$$(*) \qquad \int_{G'} P(x, u, u', u'')dG' = \int_{S'} Q(x, n_x, u, u')dS',$$

where G' is a finite (open) domain in m-dimensional Euclidean space whose boundary S' is smooth (in the sense of having at each of its points a unique and continuously turning tangent-hyperplane), x denotes a variable point, u is a function of x, u' its gradient, u'' the matrix $(\partial^2 u/\partial x_i \partial x_j)$, and n_x is the unit vector along the inward normal at x to S'. She establishes a formula corresponding to (*), and in general involving improper integrals, for a more general kind of domain G under the following assumptions: (i) for any domain G', with a smooth boundary S', such that $\overline{G'} = G' \cup S' \subseteq G$, and for any u with u'' continuous on $\overline{G'}$, the formula (*) holds, where $P\,[Q]$ is defined and continuous for $x \in \bar{G}$ and for finite u, u', u'' $[u, u'$ and for $|n_x| \leq 1]$; (ii) the boundary S of G is "almost smooth" and of finite "area"; that is, there is a closed sub-set S_s of S and a sequence $\{G_n\}$ of domains with smooth boundaries S_n such that (a) $G_1 \subseteq G_2 \subseteq \cdots \subseteq G$, (b) the area of $S_n - (S_n \cap S)$ tends to 0 as $n \to \infty$, (c) every compact set $F \subseteq G \cup (S - S_s)$ is contained in $\bar{G}_n$ for every sufficiently large n, and $\bar{G}_n \cap S_s$ is empty, and (d) the area of $S_n \cap S$ is bounded as $n \to \infty$. {In (c) the paper has G_n, but the closure $\bar{G}_n$ seems to be intended. There are also some confusing misprints in lines 28 and 33 on p. 130: for "lim" read "lim sup", and for "$S_n \subseteq \bar{U} \cap S$" read "$S \cap S_n \supseteq \bar{U} \cap S$". Remark: the author's theorem is similar to, though in some respects rather more general than, one given by O. D. Kellogg, *Foundations of potential theory* [Springer, Berlin, 1929, and Dover, New York, 1953; pp. 113–119].}

H. P. Mulholland (Exeter)

1369:

Dubovickiĭ, A. Ya. Points of complete degeneracy of the Jacobi matrix. Izv. Akad. Nauk SSSR. Ser. Mat. **22** (1958), 705–716. (Russian)

In this article a relation is established between the structure of the sets of points of complete degeneracy of the Jacobi matrix of a differentiable mapping and the degree of smoothness of the mapping.

Author's summary

1370:

Heinz, Erhard. An elementary analytic theory of the degree of mapping in n-dimensional space. J. Math. Mech. 8 (1959), 231–247.

Let E^n be the real n-dimensional Euclidean space, Λ a bounded open subset of E^n and y a continuous map of the closure $\bar{\Lambda}$ of Λ into E^n. The object of the paper is to give a new definition of Brouwer's mapping degree $d = d(y; \Lambda, z)$ for any point $z \in E^n$ which is not on the image of the boundary $\dot{\Lambda}$ of Λ, and to derive from this new definition the main properties of d (e.g., additivity, homotopy-invariance, the product theorem) and some of the main applications (e.g., Brouwer's fixed point theorem, concept and properties of the index of a mapping).

The definition is given in the following two steps. (a) Let y, in addition to having the above properties, be of class C' in Λ, i.e., have continuous first derivatives with respect to the coordinates $x_1, \cdots, x_n$ of the point $x \in \Lambda$; let z be as above, let ε_1 be the (positive) distance from z to the image of $\dot{\Lambda}$, and let ε be a positive number $< \varepsilon_1$. Let $\phi = \phi(r)$ be a continuous function defined for all non-negative r which vanishes for $\varepsilon \leq r < \infty$ and also in a small enough neighborhood of $r = 0$. Finally, ϕ is supposed to be normalized, i.e., if $|x|$ denotes the usual norm of $x \in E^n$ then the integral extended over E^n with the integrand $\phi(|x|)$ equals 1. It is then proved that

$$(1) \qquad \int_{E^n} \phi(|y(x) - z|)J[y(x)]dx,$$

where I denotes the Jacobian of the mapping $y=y(x)$, has the same value for all ϕ of the above properties. This justifies defining $d(y;\Lambda,z)$ as the integral (1). (b) It is proved from this definition that $d(y_1;\Lambda,z)=d(y_2;\Lambda,z)$ if the mappings y_1, y_2 (satisfying the assumptions of (a)) are close enough. This allows extending the definition to mappings y which are merely continuous in $\bar{\Lambda}$ by setting for $z\notin y(\dot{\Lambda})$

$$d[y;\Lambda,z]=\lim_{\nu\to\infty} d(y_\nu;\Lambda,z)$$

where $\{y_\nu\}$ ($\nu=1,2,\cdots$) is a sequence of mappings satisfying the requirements under (a) and converging to y uniformly on $\bar{\Lambda}$. *E. H. Rothe* (Ann Arbor, Mich.)

1371:

Sternberg, Shlomo. Local C^n transformations of the real line. Duke Math. J. **24** (1957), 97–102.

For $n\geqq 1$ let G^n be the group of all monotone increasing functions f of a real variable, defined in some neighborhood of the origin, of class C^n, and with $f(0)=0$, $f'(0)\neq 0$. For $n=0$, G^0 denotes the group of all continuous strictly increasing f such that $f(0)=0$. In both cases the group operation is composition of functions, and two functions are identified if they agree on some neighborhood of the origin. The problem is to find the invariants of these groups under inner automorphisms. The author solves this problem for G^0 (in terms of the fixed points of f, and the points x where $f(x)\geqq x$). For G^n he proves: If $f\in G^n$ ($n\geqq 1$) and $f'(0)=a\neq 1$, then there exists a $g\in C^{n-1}$ such that $(g^{-1}fg)(x)=ax$ for all sufficiently small $|x|$. Also, g is unique up to constant multiples. Corollary: Let F be the group of all formal power series with zero constant terms, with substitution as the group operation. Then the conjugacy classes are completely determined by the first term, provided it is not equal to one. The same corollary holds for the convergent power series.

A. Shields (Ann Arbor, Mich.)

1372:

Džrbašyan, M. M.; and Nersesyan, A. B. Some integro-differential operators and quasi-analytic classes of functions connected with them. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 5, 107–120. (Russian. Armenian summary)

For a sequence $\{\alpha_k\}$ with $0\leqq\alpha_k<1$ a sequence $D^{(k)}$ of fractional derivatives is defined; then for another sequence $\{m_n\}$ of positive numbers a class of functions $C_{m_n}(\alpha_k)$ is defined in terms of bounds imposed by these numbers on these derivatives. A criterion of Carleman type is given for such a class of functions to be quasi-analytic; that is, for the conditions $f\in C_{m_n}(\alpha_k)$ and all $D^{(k)}f(0)=0$ to imply that $f\equiv 0$. *M. M. Day* (Urbana, Ill.)

MEASURE AND INTEGRATION

See also 1425, 1561.

1373:

Mickle, E. J.; and Radó, T. On reduced Carathéodory outer measures. Rend. Circ. Mat. Palermo (2) **7** (1958), 5–33.

A real-valued set function $\Lambda(E)$ defined for all sets E of a metric separable space M is said to be a Carathéodory outer measure (C.o.m.) if $0\leqq\Lambda(E)\leqq+\infty$, $\Lambda(\emptyset)=0$, if Λ is monotone and countably subadditive, and if $\Lambda(E')+\Lambda(E'')=\Lambda(E'\cup E'')$ for all nonempty sets E', E'' having a positive distance in M (a metric outer measure in Halmos's terminology). If λ, Λ are C.o.m. and $\lambda\leqq\Lambda$, then λ is said to be a minorant of Λ. If Λ is a C.o.m. and Γ_0 any family of sets, then $\Lambda|\Gamma_0$ is the set function defined by $(\Lambda|\Gamma_0)(E)=\inf\Lambda(E-E_0)$, $E\in\Gamma_0$. If $\Lambda|\Gamma_0$ is also a C.o.m. then $\Lambda|\Gamma_0$ is said to be a reduction of Λ. Conditions are discussed in order that $\Lambda|\Gamma_0$ be a C.o.m. Given any C.o.m. Λ, the family of all sets $E\subset M$ with $\Lambda(E)=0$ (nullsets) is denoted by $\mathfrak{N}(\Lambda)$. This family plays a role in the operation of reduction. For instance: if λ is a reduction of Λ then $\lambda=\Lambda|\mathfrak{N}(\lambda)$; if $\lambda\leqq\Lambda$ are C.o.m., then $\Lambda|\mathfrak{N}(\lambda)$ is a reduction of Λ. I. If $\lambda\leqq\Lambda$ are C.o.m., then λ is a reduction of Λ if and only if every set $E\subset M$ admits of a decomposition $E=E_1\cup E_2$, $E_1\cap E_2=\emptyset$, with $\lambda(E_1)=0$, $\lambda(E_2)=\Lambda(E_2)$. Also, any reduction λ of a regular C.o.m. Λ is regular; any reduction λ of a Borel regular C.o.m. Λ is Borel regular under convenient conditions on the family Γ_0. II. If $\lambda\leqq\Lambda$ are Borel regular C.o.m., then λ is a reduction of Λ if and only if every Borel set $E\subset M$ admits of a decomposition into Borel sets $E=E_1\cup E_2$, $E_1\cap E_2=\emptyset$, with $\lambda(E_1)=0$, $\lambda(E_2)=\Lambda(E_2)$. Other interesting results are obtained and all applied to well known measures in Euclidean real spaces (in R^3 for the sake of simplicity).

The 2-dim. Hausdorff measure is denoted by H^2 and the unit sphere in R^3 by U. For every point $P\in U$ the plane through the center O of U which is orthogonal to OP is denoted by $R_2(P)$, the projection of R^3 into $R_2(P)$ by T_P, the 2-dim. Lebesgue measure in $R_2(P)$ by L_2. Also, $\mathfrak{Z}$ is the family of all sets $Z\subset U$ with $H^2(Z)=0$, $\mathfrak{F}_0{}^*$ is the family of all Borel sets $B\subset R^3$ with $L_2T_PB=0$ for all $P\in U$, $\mathfrak{F}_0$ is the family of all Borel sets $B\subset R^3$ for which there is some $Z_B\in\mathfrak{Z}$ such that $L_2T_PB=0$ for all $P\in U-Z_B$. A set function φ in R^3 is said to satisfy the strong [weak] projection condition if $L_2T_PE\leqq\varphi(E)$ for all sets $E\subset R^3$ and all $P\in U$ [all $P\in U-Z_E$ and some $Z_E\in\mathfrak{Z}$]. III. $H^2|\mathfrak{F}_0{}^*$ $[H^2|\mathfrak{F}_0]$ is the largest Borel regular minorant of H^2 whose nullsets are all the sets of $\mathfrak{F}_0{}^*$ $[\mathfrak{F}_0]$. IV. $H^2|\mathfrak{F}_0{}^*$ $[H^2|\mathfrak{F}_0]$ satisfies the strong [weak] projection condition. For any Borel set $B\subset R^3$, let $B=\bigcup b_i$ be any countable decomposition of B into disjoint Borel sets b_i and let $\mu^*(B)=\sup\sum_i(\sup L_2T_Pb_i)$, where each of the interior suprema is taken with respect to all $P\in U$, and the exterior supremum is taken with respect to all decompositions $B=\bigcup b_i$. V. μ^* is the smallest Borel regular C.o.m. in R^3 satisfying the strong projection condition. If the interior supremum above is taken with respect to all $P\in U-Z$, where Z is any fixed set $Z\in\mathfrak{Z}$, then we have a set function μ_Z, and then by $\mu(E)$ is denoted the new set function defined by $\mu(E)=\inf\mu_Z(E)$, $Z\in\mathfrak{Z}$. VI. μ is the smallest Borel regular C.o.m. in R^3 satisfying the weak projection condition. Concerning the so-called integral-geometrical measure I^2 in R^3 a number of conjectures are made. *L. Cesari* (Baltimore, Md.)

1374:

Dowker, Yael Naim; and Erdős, Paul. Some examples in ergodic theory. Proc. London Math. Soc. (3) **9** (1959), 227–241.

Let (Y,γ,m) be a σ-finite, non-atomic measure space

and T (or T_i, $i=1, 2$) a 1-1 ergodic measure preserving transformation of Y onto itself. Let

$$M_n(T, A, y) = \sum_{i=0}^{n-1} f_A(T^i y)/n,$$

where f_A is the characteristic function of A.

Among the several counter examples which are given are the following:

(1) Let Y be $[0, 1)$, m be Lebesgue measure and let the system be conservative. Let $\{c_j\}$ satisfy $c_j \geq 0$ and $\sum_{j=1}^{\infty} c_j = \infty$. Then there exists a bounded measurable function f on Y such that $\int f dm = 0$ and $\sum_{j=1}^{\infty} c_j f(T^j x)$ does not converge on any subset of positive measure. This answers a question raised by Halmos [Ann. of Math. (2) **48** (1947), 735–754; MR **9**, 137] and by Standish [Pacific J. Math. **6** (1956) 553–564; MR **18**, 479].

(2) Even if $T_2 = T_1^{-1}$, the limit as $n \to \infty$ of $M_n(T_2, A, y)/M_n(T_1, A, y)$ need not exist almost everywhere when $m(Y) = \infty$. Also even though

$$\lim_{n\to\infty} M_n(T_1, A, y) = \lim_{n\to\infty} M_n(T_2, A, y) = 0$$

for sets A with $0 < m(A) < \infty$, it is shown that $M_n(T_1, A, y) - M_n(T_2, A, y)$ need not converge to zero for all measurable sets A.

(3) There exist measure preserving transformations T_i on $[0, \infty)$, $i = 1, 2$, such that

$$\limsup_{n\to\infty} \sum_{j=0}^{n-1} f(T_1{}^j y) \Big/ \sum_{j=0}^{\infty} f(T_2{}^j y) = \infty$$

for almost all y and any integrable f with $\int f dm \neq 0$. This provides a counter example to Theorem 3 in the case $m(Y) = \infty$ in a paper by the reviewer [Pacific J. Math. **5** (1955), 869–876; **6** (1956), 795; MR **17**, 833; **18**, 882; see p. 874].
P. Civin (Eugene, Ore.)

1375:

Nakanishi, Shizu. L'intégrale de Denjoy et l'intégration au moyen des espaces rangés. IV. Proc. Japan Acad. **34** (1958), 96–101.

Suite aux notes précédentes [même Proc. **32** (1956), 678–683; **33** (1957), 13–18, 265–270; MR **19**, 256, 1167]; donne une nouvelle définition de l'intégrale de Denjoy au moyen des espaces rangés.
A. Appert (Angers)

1376:

Demers, Maurice R.; and Federer, Herbert. On Lebesgue area. II. Trans. Amer. Math. Soc. **90** (1959), 499–522.

In der vorangegangenen Arbeit [Ann. of Math. (2) **61** (1955), 289–353; MR **16**, 683] hatte der zweite der beiden Verfasser wesentliche Teile der Theorie des Lebesgueschen Flächeninhaltes, insbesondere die Cesarische Ungleichung, verallgemeinert auf den Fall einer stetigen Abbildung f, die nicht notwendig in einer Zelle, sondern einer beliebigen endlich triangulierbaren Teilmenge X der Ebene E_2 definiert ist und deren Werte nicht notwendig in E_3, sondern in einem beliebigen Raum E_n liegen. Gewisse Sätze ließen sich sogar für Abbildungen beweisen, die in einer k-dimensionalen endlich triangulierbaren Teilmenge X von E_k definiert sind. Die vorliegende Arbeit dehnt nun diese Resultate aus auf den Fall, daß X nicht notwendig in E_k (oder eine k-dimensionale Mannigfaltigkeit) eingebettet, sondern ein beliebiger k-dimensionaler endlich triangulierbarer Raum ist. Dies wird ermöglicht durch einen weitgehenden Ausbau der in der früheren Arbeit entwickelten Methoden: Gebrauch der Kohomologietheorie, insbesondere Definition der Norm von ganzzahligen Čechschen Kohomologieklassen, darauf gestützte dimensionstheoretische Betrachtungen einschließlich Definition einer Multiplizitätsfunktion M für Abbildungen f nebst Sätzen über simpliziale Approximationen und schließlich Betrachtungen über den Zusammenhang zwischen der k-dimensionalen Stabilität einer Abbildung und der $(k-1)$-dimensionalen Stabilität ihrer "Schichten" (vorher nur im Fall $k=2$ ausgeführt). Die Cesarische Ungleichung $L_k(f) \leq \sum L_k(P^\xi \circ f)$, in der ξ alle strikt monoton wachsenden Folgen $(\xi_1, \cdots, \xi_k)$ mit ganzzahligen ξ_i zwischen 1 und n einschließlich durchläuft und P^ξ die Projektion von E_n auf den von den $\xi_1, \cdots, \xi_k$-ten Koordinatenachsen aufgespannten Unterraum sowie L_k den k-dimensionalen Lebesgueschen Inhalt bedeuten, wird in den folgenden Fällen bewiesen: $f(X)$ hat das $(k+1)$-dimensionale Hausdorffsche Maß 0 oder es ist $k=2$. Im Fall $k=n$ wird $L_k(f)$ das über E_k erstreckte Integral der Multiplizitätsfunktion M hinsichtlich des k-dimensionalen Lebesgueschen Maßes.
K. Krickeberg (Heidelberg)

FUNCTIONS OF A COMPLEX VARIABLE

See also 1295, 1398, 1399, 1451, 1529.

1377:

Künzi, Hans P. Quasikonforme Abbildungen. Ann. Acad. Sci. Fenn. Ser. A. I, no. 249/2 (1958). 24 pp.

The paper is an expository account of the developments in the theory of quasi-conformal mapping. The author discusses the analytic and geometric definitions of quasi-conformality, the elementary distortion theorems, the properties of locally quasi-conformal functions, and extremal quasi-conformal mappings. Various present-day developments, such as the theory of elliptic partial differential equations, are mentioned, and the paper contains an extensive bibliography of the important contributions to the field.
A. J. Lohwater (Ann Arbor, Mich.)

1378:

Nevanlinna, Rolf. Über fastkonforme Abbildungen. Ann. Acad. Sci. Fenn. Ser. A. I. no. 251/7 (1958). 10 pp.

Let $y = y(x)$ be a smooth complex valued function of the complex variable x, and let $E_{yx} = 2i\partial y/\partial x$. Then the vanishing of E_{yx} is equivalent to the analyticity of y. The author considers the formula

$$(*)\quad 2\pi i y(t) = \int y(x)(x-t)^{-1}dx - \iint E_{yx}(x)(x-t)^{-1}d\sigma_x,$$

and discusses consequences to be obtained by imposing various bounds on E_{yx}. For example, let $y(x)$ be smooth for $0 \leq |x| \leq R$, and satisfy (A) $\liminf rM(r) = 0$, where $M(r) = \max_{|x|=r} |y(x)|$, and (B) $\int_0 E_r dr < \infty$, where $E_r = \max_{r \leq |x|} |E_{yx}(x)|$. Then y is continuous at $x=0$, and $y(0)$ is given by (*).
H. L. Royden (Zürich)

1379:

Nakai, Mitsuru. On a ring isomorphism induced by quasiconformal mappings. Nagoya Math. J. **14** (1959), 201–221.

Let R be a Riemann surface, and let BD be the space of smooth bounded complex-valued functions on R with a finite Dirichlet integral. Define $\|f\|$ as sup $|f(p)|+[D(f)]^{1/2}$, where D denotes the Dirichlet integral. Let $M(R)$ be the completion of BD in this norm. Then $M(R)$ is a ring, and a ring isomorphism φ between the rings $M(R)$ and $M(R')$ is called normal if φ and φ^{-1} take (a) sequences uniformly convergent on compact sets into sequences uniformly convergent on compact sets and (b) sequences convergent in the sense of the Dirichlet integral into sequences convergent in the sense of the Dirichlet integral.

The author characterizes the rings $M(R)$ in terms of Tonelli absolute continuity and shows that each quasiconformal mapping of R onto R' induces a normal isomorphism of $M(R)$ onto $M(R')$ and that, conversely, each normal isomorphism of $M(R)$ onto $M(R')$ is induced by a quasi-conformal mapping of R onto R'.

H. L. Royden (Zürich)

1380:

Derwidué, L. Sur des équations obtenues à partir de déterminants. Mathesis **67** (1958), 214–217.

Let the sequence of polynomials f_j and k_j have the form

$$f_j = x+q_jx^3+p_jx^5+\cdots+a_jx^{2r_j+1},$$
$$k_j = U_j+Q_jx^2+P_jx^4+\cdots+A_jx^{2s_j} \quad (j=0,1,2,\cdots),$$

where all the coefficients are real. Consider the determinental equation

$$H_n(x) = \begin{vmatrix} f_0+k_0i+1 & -1 & & & \\ 1 & c_1f_1+k_1i & -1 & & \\ & 1 & c_2f_2+k_2i & -1 & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 1 & c_{n-1}f_{n-1}+k_{n-1}i \end{vmatrix} = 0,$$

in which the places not filled in are zero, and the c_i are real, $i=\sqrt{-1}$. The author proves the following result. Let ν_j be the number of roots of $f_j(x)=0$ $(j=1,2,\cdots,n-1)$ having positive real parts, and ν_0 those of the polynomial $f_0+k_0i+1=0$. If no $c_j=0$ and if μ of the c_j are negative, the number of roots of $H_n=0$, with positive real parts, is precisely $\nu_0+\sum_{j=1}^{n-1}\nu_j+\mu$. This result generalizes one due to the author in his book reviewed as #1284 above, pp. 403–414.

I. A. Barnett (Cincinnati, Ohio)

1381:

Turán, P. A remark concerning the behaviour of a power-series on the periphery of its convergence-circle. Acad. Serbe Sci. Publ. Inst. Math. **12** (1958), 19–26.

Let D be a region in the plane, and $A(D)$ the algebra of functions f that are analytic in D. If ϕ is a conformal map of D onto itself, then $g(z)=f(\phi(z))$ is in $A(D)$ for any $f\in A(D)$. It is natural to say that f and g are equivalent in this case, and to ask for other properties that must of necessity be shared by f and g. In the present paper, the author looks at this general question for D the unit circle. Here, it is natural to look for properties associated with the behavior of power series on the boundary of the unit circle. It is easily seen that if the power series for f is Abel summable at a boundary point ζ, then the power series for g is Abel summable at $\phi(\zeta)$. In an earlier paper, the author had conjectured that this was also true for convergence [Eötvös L. Tud.-Egy. Kiadv. Term.-Tud. Kar Évk. **1952–53**, 5–13; MR **17**, 598]. In the present paper, he shows that this is false.

R. C. Buck (Los Angeles, Calif.)

1382:

Alpár, László. Remarque sur la sommabilité des séries de Taylor sur leurs cercles de convergence. I. Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), no. 1/2, 1–12. (Hungarian and Russian summaries)

This is an extension of the paper by Turán reviewed above. The author shows that if the power series for f is summable at ζ by Cesàro kth means (C,k), it need not necessarily be the case that the related function g have a (C,k) summable power series at the corresponding boundary point $\phi(\zeta)$.

R. C. Buck (Los Angeles, Calif.)

1383:

Bajšanski, Bogdan. Généralisation d'un théorème de Carlemann. Acad. Serbe Sci. Publ. Inst. Math. **12** (1958), 101–108.

This is a continuation of the preceding two papers; the notation is the same. Carlemann had shown [Ark. Math. Fys. **15** (1920), 1–13] that if ϕ was analytic on the closed unit circle, and if the image of the boundary $\phi(e^{i\theta})$ lay inside the unit circle, touching it only at $z=1$ with first order contact there, then $f(\phi(z))$ will have a power series convergent at $z=1$ if f does. The author generalizes this result by replacing the last restriction on ϕ by one which requires that $\phi(z)=z^c+Ai^p(z-1)^p\{1+o(1)\}$ as $z\to 1$, where $\operatorname{Re}(A)\neq 0$. The proof depends upon a refinement of a previous theorem of the author [Acad. Serb. Sci. Publ. Inst. Math. **10** (1956), 131–152; MR **18**, 888] and a delicate estimate of contour integrals.

R. C. Buck (Ventura, Calif.)

1384:

★Bauer, Friedrich L. The quotient-difference and epsilon algorithms. On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 361–370. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U.S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) $4.50.

This paper is concerned with formulas relating the Stieltjes continued fraction to its power series expansion, obtained by writing the continued fraction in the form $s_0/z-g_0(c-g_1)/1-g_1(1-g_2)/z-g_2(c-g_3)/1-\cdots$ and considering even and odd parts.

H. S. Wall (Austin, Tex.)

1385:

Krzyż, J. A symmetrization result for maximum modulus. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 557–559.

The author proves the following theorem. Let W be a simply connected bounded Riemann Surface over the w-plane which is the biuniform map of the circle $|z|<1$ by $f(z)$. Let $f(z)$ be regular in $|z|<1+\delta$ $(\delta>0)$ and normalized

so that $f(0)=0$, $f'(0)>0$, and $f(z)\neq 0$ for $0<|z|\leq 1$. Let W^* be a simply connected Riemann surface obtained by circular symmetrization of W with respect to the positive real axis and let $f^*(z)$ be the symmetrized function mapping the unit circle onto W^* such that $f^*(0)=0$ and $f^{*\prime}(0)>0$. If $w_0=f(z_0)\neq 0$, then there exists a positive z_0^* such that $|w_0|=f^*(z_0^*)$ and $z_0^*<|z_0|$.

G. Springer (Lawrence, Kans.)

1386:

Epstein, Bernard. The kernel-function and conformal invariants. J. Math. Mech. 7 (1958), 925–936.

Certain conformal invariants closely related to the Bergman kernel-function are considered. It is shown that such invariants can be used to determine the conformal modulus of a doubly-connected domain. Two theorems are given of which the proofs are of tedious computational nature.

Y. Komatu (Tokyo)

1387:

Kung, Sun. On the Bloch constant for multiply connected regions. Acta Math. Sinica 7 (1957), 513–519. (Chinese. English summary)

Let D be a bounded multiply-connected domain of the z-plane. Let the Gauss curvature of the Bergman metric $ds^2=T|dz|^2$, where $T=\partial^2 \ln K_D/\partial z\partial\bar{z}$ (K_D being the Bergman kernel-function for the domain D), attain its minimum τ_a at the point $a\in D$, and let $w=f(z)$ be a function holomorphic in D and satisfying the condition $f'(b)=1$ (with $b\in D$) and mapping D onto a domain D_f of the w-plane. It is proved that the region D_f contains a circle of radius $(-8\tau_a T(b)/3)^{-1/2}$. Further, if the function $w=f(z)$ is univalent in D, then the domain D_f contains a circle of radius $(2\tau_a T(b))^{-1/2}$.

B. A. Fuks (RŽ Mat **1959** #296)

1388: X 1958 ed

★**Thinius, E. Ortskurvenlehre und konforme Abbildungen in der komplexen Ebene.** R. v. Decker's Verlag; G. Schenck; Hamburg-Berlin-Bonn, 1959. 98 pp. DM 19.80; \$4.71.

Das Buch gibt eine Einführung in das graphische Rechnen mit komplexen Zahlen mit Anwendungen auf die Theorie der Wechselstromschaltungen. Untersucht wird vor allem die konforme Abbildung von Ortskurven (bezifferten Kurven in der komplexen Zahlenebene) durch lineare, quadratische und elementartranszendente Funktionen. Die Darstellung ist an einigen Stellen unklar und nicht ganz korrekt.

F. Stallman (Giessen)

1389:

Schwartz, Marie-Hélène. Remarques et exemples relatifs à des applications simpliciales et paraboliques. Ann. Acad. Sci. Fenn. Ser. A. I. no. **250/33** (1958), 14 pp.

The author considers simplicial mappings from one manifold into another. Such a mapping is called parabolic relative to an exhaustion of the first manifold if it satisfies certain conditions which say in effect that each point on the second manifold is covered the same number of times on the average. Various types of defect are defined for such mappings and relations between them are established.

H. L. Royden (Zürich)

1390:

Lehner, Joseph. On modular forms of negative dimension. Michigan Math. J. 6 (1959), 71–88.

Define $e(z)=e^{2\pi iz}$. Let V run through a subset of the matrices $\begin{pmatrix} ab\\ cd\end{pmatrix}$ of the modular group $\Gamma(1)$, such that each second row occurs once and only once (for the purpose on hand the first row is immaterial, as long as $ad-bc=1$). Then, if $s>2$,

$$F_\mu(\tau)=\tfrac{1}{2}\sum_V e(-(\mu-\alpha)V\tau)\varepsilon(V)^{-1}(-i(c\tau+d))^r$$

represents a modular form of real negative dimension $r=-s<-2$. Here μ is a positive integer, $0<\alpha<1$ and $\varepsilon(V)$ is a multiplier system for the dimension r. The series converges absolutely and uniformly in every region $\Im\tau\geq y_0>0$, whence the regularity of $F_\mu(\tau)$ in $\Im\tau>0$ and its modular character readily follow. If $s=2$, the series is no longer absolutely convergent. Using an idea of Hecke, Petersson [Math. Ann. **103** (1930), 369–436] introduced a convergence factor $|c\tau+d|^{-\sigma}$ and, letting $\sigma\to 0$, obtained modular forms of dimension -2. In the present paper the author shows that if one sums the series in a certain prescribed order, then it converges to an entire modular form of dimension -2. For that, one defines

$$(*)\qquad F_\mu(\tau)=e(-(\mu-\alpha)\tau)+H(\tau),$$

with

$$H(\tau)=\lim_{K\to\infty}\frac{1}{2}\sum_{k=-K}^{K}\sum_{m=-K}^{K}{}' e(-(\mu-\alpha)V_{k,-m}\tau)\times\varepsilon(V_{k,-m})^{-1}(-i(k\tau-m))^r$$
$$=\lim_{K\to\infty}\sum_{k=1}^{K}\sum_{m=-K}^{K}{}' e(-(\mu-\alpha)V_{k,-m}\tau)\times(V_{k,-m})^{-1}(-i(k\tau-m))^r,$$

with $V_{k,-m}=\begin{pmatrix} m' & k'\\ k & -m\end{pmatrix}$, $mm'\equiv -1 \pmod k$, $0<m'<k$ and the inner summation extended only over m prime to k. In the proof of the convergence, use is made of the Assumption A: Define

$$A_{k,\mu}(m)=\sum_{\substack{h=0\\(h,k)=1}}^{k-1}\varepsilon^{-1}(V_{k,-h})e[-(\mu-\alpha)h'+(m+\alpha)h/k];$$

then

$$A_{k,\mu}(m)\leq C_\varepsilon(\rho m+\sigma,k)^{1/2}k^{1/2+\varepsilon}$$

for $k\geq 1$ and every $\varepsilon>0$, unless $\alpha=0$, $m=0$, when $A_{k,\mu}(0)\leq C$. For $s>3/2$, and assuming A, the author shows: (i) $F_\mu(\tau)$ is regular in $\Im\tau>0$; (ii) for $W\in\Gamma(1)$, $W=\begin{pmatrix}\cdots\\ \gamma\delta\end{pmatrix}$, $F_\mu(W\tau)=\varepsilon(W)(-i(\gamma\tau+\delta))^s F_\mu(\tau)$; (iii) $F_\mu(\tau)=e(-(\mu-\alpha)\tau)+\sum_0^\infty a_m e((m+\alpha)\tau)$, with

$$(**)\quad a_m=2\pi\sum_{k=1}^{\infty}k^{-1}A_{k,\mu}(m)\left(\frac{\mu-\alpha}{m+\alpha}\right)^{1/2(r+1)}\times I_{-r-1}(4\pi k^{-1}(\mu-\alpha)^{1/2}(m+\alpha)^{1/2}),$$

where $I_l(z)$ stands for the Bessel function of imaginary argument. Assumption A, however, is proven (by reducing the sums to Kloosterman sums) only for $s=2$. It would be of considerable interest to prove it for $s<2$, as (*) would then represent modular forms of dimension r, $-2<r<-3/2$ and (**) would represent their Fourier

coefficients by convergent series (at present no such representations are known). The paper finishes by showing that every modular form $G(\tau)$ of dimension -2 may be written as $\sum_{\nu=1}^{\mu} b_\nu F_\nu(\tau)+K(\tau)$, with constant b_ν's and $K(\tau)$ a cusp form. *E. Grosswald* (Princeton, N.J.)

1391:

Džrbašyan, M. M. Development of meromorphic functions in a generalized Maclaurin's series. Dokl. Akad. Nauk SSSR **125** (1959), 707–710. (Russian)

Es sei $\{a_k\}$ ($|a_k|>1$, $k\geq 0$, $\lim|a_k|=\infty$) eine Folge komplexer Zahlen, geordnet nach wachsendem Betrag; $\alpha_k=1/\bar{a}_k$. Es sei $\phi(z)$ eine meromorphe Funktion, die höchstens in den Punkten a_k Pole hat, die Vielfachheit des Poles im Punkt a_r sei höchstens gleich der Anzahl der Elemente a_r in der Folge $\{a_k\}$. Die Entwicklung $\sum_0^\infty c_n\varphi_n(z)$ von $\phi(z)$ nach dem auf $|z|=1$ orthonormalen Funktionensystem

$$\varphi_n(z)=\left(\frac{1-|\alpha_n|^2}{2\pi}\right)^{1/2}\prod_{k=0}^{n-1}\frac{z-\alpha_k}{1-\bar{\alpha}_k z}$$

mit

$$c_n=\int_{|\zeta|=1}\phi(\zeta)\overline{\varphi_n(\zeta)}|d\zeta|$$

konvergiert gegen $\phi(z)$ gleichmässig und absolut auf jedem beschränkten abgeschlossenen Bereich, der keinen Punkt a_k enthält. Ist $a_k\equiv\infty$, so ist $\varphi_n(z)=z^n$ und $\sum_0^\infty c_n\varphi_n(z)$ ist die MacLaurinsche Entwicklung von $\phi(z)$.

F. Huckemann (Giessen)

1392:

Hervé, Michel. Valeurs exceptionnelles d'une fonction méromorphe au voisinage d'un ensemble singulier de capacité nulle. Ann. Acad. Sci. Fenn. Ser. A. I. no. 250/14 (1958), 4 pp.

Let E be a compact set of capacity zero in the z-plane and D be a domain containing E in its interior. Let $w=f(z)$ be a single-valued meromorphic function in the domain $D-E$ which has an essential singularity at every point of E. A value α is called exceptional at a point ζ of E, if there is a neighborhood of ζ in which $f(z)$ omits the value α. Let Ξ be a relatively closed set in D which cannot be represented as the union of two non-empty disjoint sets closed in D. Then, for simplicity, Ξ is called a continuum contained in D. Let X be a non-degenerate continuum in the w-plane. Then, a continuum Ξ contained in D is called an inverse image of X by the function $w=f(z)$, provided that the following three conditions are satisfied: (1) $z\in\Xi$ implies $z\in E$ or $f(z)\in X$; (2) Ξ is not degenerate to a point of E; (3) every continuum containing Ξ and satisfying the condition (1) is Ξ itself. Applying his earlier result [J. Math. Pures Appl. **35** (1956), 161–173; MR **18**, 385, 169], the author proves: If $f(z)$ admits two exceptional values w_1, w_2 at a point ζ of E and at least one other exceptional value at the point ζ (this assumes that the classical form of Picard's theorem does not hold in the present case), then, for a given non-degenerate continuum X, such that w_1 and w_2 belong to the same connected component of the complement CX of X, with respect to the w-plane, every neighborhood of ζ contains an inverse image of X. This is a remarkable result closely related to a conjecture on a compact set of essential singularities of capacity zero [cf. the author's paper cited above; Lehto, Ann. Acad. Sci. Fenn. Ser. A. I. no. **249/3** (1957); MR **20** #3282; Lohwater, ibid. no. **250/22** (1958); MR **20** #3988; the reviewer, Amer. Math. Soc. Transl. (2) **8** (1958), 1–12; MR **19**, 1171].

K. Noshiro (Nagoya)

1393:

Seidel, W. Holomorphic functions with spiral asymptotic paths. Nagoya Math. J. **14** (1959), 159–171.

Let S be a simple continuous curve $z=\zeta(t)$ $(0\leq t<\infty)$ such that $0<|\zeta(t)|<1$, $\lim_{t\to\infty}|\zeta(t)|=1$, $\lim_{t\to\infty}\arg\zeta(t)=\infty$. Then, S is called a spiral in $|z|<1$ which approaches $|z|=1$ asymptotically. For any value of t, starting with the point $\zeta(t)$ on the curve, describe the curve in the sense of increasing t and let t' be the first value of t for which $\arg\zeta(t')=\arg\zeta(t)+2\pi$. Denoting by $\rho(\zeta(t),\zeta(t'))$ the non-Euclidean (n-E) distance between these two points, the author introduces the following measures for the "tightness" of a spiral S:

$$\mu(S)=\liminf_{t\to\infty}\rho(\zeta(t),\zeta(t')),\quad \bar{\mu}(S)=\limsup_{t\to\infty}\rho(\zeta(t),\zeta(t')).$$

Using the theory of normal families, the author proves: (1) Let $w=f(z)$ be an unbounded holomorphic function in $|z|<1$, with the property that it remains bounded on some spiral S with $\mu(S)<\infty$ which approaches $|z|=1$ asymptotically. Then, there exists a spiral S' in $|z|<1$, such that, denoting by Σ the union of all n-E open circular discs with n-E centers on S' and of fixed n-E radius, the complement of the range $R(f,\Sigma)$ with respect to $|w|<\infty$ consists of at most one point. (2) If $f(z)$ is holomorphic in $|z|<1$ and if $S:z=\zeta(t)$, $0\leq t<\infty$, is a spiral for which $\bar{\mu}(S)<\infty$ and $\lim_{t\to\infty}|f(\zeta(t))|=\infty$, then the same conclusion as in (1) holds. (3) There exists a function $\Psi(z)$, holomorphic in $|z|<1$, whose maximum modulus tends to infinity as slowly as one wishes, with the property that at almost all $e^{i\theta}$, the complement of the range $R(\Psi,\Delta(\theta))$ with respect to $|w|<\infty$ consists of at most one point for every Stolz angle $\Delta(\theta)$ with vertex $e^{i\theta}$.

K. Noshiro (Nagoya)

1394:

Sunouchi, Gen-ichirô. On functions regular in a half-plane. Tôhoku Math. J. (2) **9** (1957), 37–44.

This paper contains the statements and proofs of several theorems connected with the function g^* related to functions of the class H_p (of the upper half plane). In fact, if $\varphi\in H_p$, the author defines

$$g_\alpha^*(x)=\left\{\frac{1}{\pi}\int_0^\infty y^{2\alpha}dy\int_{-\infty}^{+\infty}\frac{|\varphi'(t+iy)|^2}{|t-z|^{2\alpha}}\,dt\right\}^{1/2}.$$

This is in analogy with a similar function for the unit circle considered, e.g., by A. Zygmund [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 208–212; MR **17**, 1080]. The author proves

$$\int_{-\infty}^{+\infty}[g_\alpha^*(x)]^p dx\leq A_{p,\alpha}\int_{-\infty}^{+\infty}|\varphi(x)|^p dx,$$

$\alpha>1/p$ for $0\leq p\leq 2$, and $\alpha>\frac{1}{2}$ for $p\geq 2$.

Other results are deduced in strict analogy with the case of the unit circle. *E. M. Stein* (Chicago, Ill.)

1395:

Dundučenko, L. O.; and Kas'yanyuk, S. A. On analytic functions bounded in n-tuply connected circular

regions. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 111–115. (Ukrainian. Russian and English summaries)

Proceeding from the results of V. A. Zmorovich [same Dopovidi **1958**, 489–492; MR **20** #5277], the author establishes a structural formula for a class of functions, limited in their modulus in the vicinity of the boundary of K_n and analytic (regular or meromorphic) in an n-connected circular region K_n. A series of exact estimates were obtained in classes of limited regular functions. Exact estimates were also obtained for the expressions $|f'(z)|$ and $\operatorname{Re} f(z)$ in the class $C(K_n)$ of functions regular in K_n and possessing a positive real part.

Authors' summary

1396:

Leont'ev, A. F. On the completeness of the system $\{z^{\lambda_n}\}$ on curves in the complex plane. Dokl. Akad. Nauk SSSR **121** (1958), 797–800. (Russian)

Voici l'énoncé (peut-être un peu moins général que l'énoncé original de l'auteur, compliqué quant à la description de la courbe L) du théorème essentiel: soit L l'ensemble composé de m courbes issues de l'origine tendant vers l'infini, qui ne se coupent pas, chacune sans point double, la partie de chacune de ces courbes, contenue dans une partie finie du plan, étant rectifiable; et soient $G_1, G_2, \cdots, G_m$ les domaines du plan compris respectivement entre deux branches de L. Supposons que chaque G_j contienne un angle d'ouverture π/α_j. Supposons, qu'en désignant par $\sigma(z)$ la longueur d'une branche de L, on ait $d\sigma(z) \leqq M d|z|$. Soit $p(z)$ une fonction réelle, continue définie sur L, telle que

$$p(z) \geqq p_0(|z|) = p_0(a) + \int_a^{|z|} \frac{\omega(t)}{t}\, dt,$$

pour $|z|$ assez grand, où $\omega(t) \geqq 0$, $\omega(t) \uparrow \infty$; et soit, enfin, $\{\lambda_n\}$ une suite d'entiers positifs croissants, qui possède une densité égale à $1-\sigma$. Si $\pi/\alpha_j \geqq 2\pi\sigma$ $(j=1, 2, \cdots, m)$, et si, pour un $\omega > \max(\beta_1, \cdots, \beta_m)$, $\pi/\beta_j = \pi/\alpha_j - 2\pi\sigma$,

$$\int^{\infty} P_0(r)/(r' + \omega)dz = \infty,$$

la suite $\{z^{\lambda_n}\}$ est complète sur L dans le sens suivant: $Q(z)$ étant une combinaison linéaire quelconque (finie) des z^{λ_n} on a:

$$\inf_Q \int_L e^{-p(z)} |f(z) - Q(z)|^2 d\sigma = 0,$$

lorsque $\int_L e^{-p(z)} |f(z)|^2 d\sigma < \infty$. L'auteur généralise un théorème de Mandelbrojt [*Séries adhérentes, régularisation des suites, applications*, Gauthier-Villars, Paris, 1952; MR **14**, 542], en montrant que dans les mêmes conditions la suite $\{1, z^{\lambda_n}\}$ est complète dans le sens

$$\inf_Q \max_{z \in L} e^{-p(z)} |f(z) - Q(z)| = 0$$

lorsque $e^{-p(z)} f(z) \to 0$ $(z \to \infty, z \in L)$.

S. Mandelbrojt (Paris)

1397:

Chen, Kien-kwong. Generalizations of Minkowski's inequality with applications to the theory of mean approximation by integral functions. Sci. Record (N.S.) **2** (1958), 81–85.

While in a previous paper [same Record **1** (1957), 19–23; MR **20** #3408] uniform approximation by integral functions of order ρ is considered, here mean approximation is treated. Let $\varphi(t)$ $(t>0)$ be non-negative and satisfy certain conditions, which include the case $\varphi(t) = t^q$, $q > 1$; let $p(z) \geqq 0$, and the norm of $f(z)$ $(z = x + iy)$ on some domain K_ρ be

$$\|f(z)\|_\varphi = \varphi^{-1}\left\{ \iint_{K_\rho} p(z) \varphi(|f(z)|) dx dy \right\};$$

and define the φ-modulus of continuity of $f(z)$ on K_ρ with weight $p(z)$ as $\omega(\delta; f, K_\rho) = \max_h \|f(z+h) - f(z)\|_\varphi (h \leqq \delta)$. Then a number of theorems are stated, without proof, about approximation to $f(z)$ by integral functions $g_\sigma(z)$ of finite types σ of the order ρ on various domains K_ρ, including the real axis and the strip; for instance:

Theorem 1: If $K_1 = (-\infty, \infty)$, $f(x) = O(|x|^q)$ $(|x| \to \infty)$, and $\omega(\delta; f, K_1)_\varphi \leqq \Omega(\delta)$, then there are g_σ's $(\rho = 1; \delta \to \infty)$ such that $\|f(z) - g\sigma(z)\|_\varphi \leqq c(\Omega)(\sigma^{-1})$, where c is a constant.

H. Kober (Birmingham)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

See also 1585, 1630.

1398:

Hitotumatu, Sin. On quasi-conformal functions of several complex variables. J. Math. Mech. **8** (1959), 77–94.

After reviewing briefly various definitions of quasi-conformal mappings in two and more (real) dimensions, the author presents a new definition which is applicable to the space C_n of n complex variables (which is not to be confused with the space of $2n$ real variables). The basic idea is that of a "discwise quasi-conformal function with dilatation K", which is a function of class C^1 in a domain D of C_n which is K-quasi-conformal as a function of the single complex variable t in the intersection of D with every "holomorphic plane" $z_i = a_i t + b_i$ $(i = 1, 2, \cdots, n)$. A "K-D.Q.C." (K-discwise quasi-conformal) mapping is one defined by n functions of the aforementioned type. The author concludes with a brief analysis of some properties of such mappings (restricted to the case $n=2$). It turns out that these mappings lack many properties that one would desire to carry over from the case $n=1$. Also, it is pointed out that the present definition is inconsistent with one recently given by S. Bergman.

Bernard Epstein (Philadelphia, Pa.)

1399:

Bergman, Stefan. A class of quasi-pseudo-conformal transformations in the theory of functions of two complex variables. J. Math. Mech. **7** (1958), 937–956.

In the spaces of two complex variables the author considers quasi-pseudo-conformal transformations which change the hypersphere into a Reinhardt circular domain. The main purpose is to obtain bounds for the distortion of the non-euclidean line element by a mapping of this type under certain additional conditions.

Y. Komatu (Tokyo)

1400:

Look, C. H.; and Chung, T. D. An extension of Privalof's theorem. Acta Math. Sinica **7** (1957), 144–165. (Chinese. English summary)

Let $m=2n$ $(n\geqq 2)$, $z_\alpha=u_\alpha+iu_{n+\alpha}$ $(\alpha=1,\cdots,n)$, and $\mathscr{D}$ be a domain of the $2n$-dimensional space of $u_1,\cdots,u_{2n}$, and its boundary Ω be a $(2n-1)$-dimensional smooth orientable manifold of class C^2 defined by

$$F(u_1,\cdots,u_{2n})=0.$$

Let $K_{(2n-1)}(z,\zeta)$ be a complex exterior differential form of degree $2n-1$

$$K_{(2n-1)}(z,\zeta)=\sigma\sum_{\alpha=1}^{n}(-1)^{\alpha}\frac{\partial}{\partial z_\alpha}\frac{1}{[r(z-\zeta)]^{2n-2}}$$
$$\times dz_1\cdots dz_n\,\overline{dz}_1\cdots\overline{dz}_{\alpha-1}\,\overline{dz}_{\alpha+1}\cdots\overline{dz}_n,$$

where r $(z-\zeta)$ denotes the euclidean distance of $z=(z_1,\cdots,z_n)$ and $\zeta=(\zeta_1,\cdots,\zeta_n)$, and $\sigma=(n-2)!/(2\pi i)^n$.

We then obtain the theorem: If $f(z)$ is a continuous function of complex value defined on Ω, which satisfies a Hölder condition and defines a function $F(w)$ in $\mathscr{D}$ such that

$$F(w)=\int_\Omega f(z)K_{(2n-1)}(z,w),$$

and if w_0 is an arbitrary point on Ω, then we have

$$F_i(w_0)=\text{V. P.}\int_\Omega f(z)K_{(2n-1)}(z,w_0)+\frac{1}{2}f(w_0),$$

$$F_e(w_0)=\text{V. P.}\int_\Omega f(z)K_{(2n-1)}(z,w_0)-\frac{1}{2}f(w_0),$$

where $F_i(w_0)$ and $F_e(w_0)$ denote the limit values of $F(w)$ when w approaches w_0 from the inner part and the outer part of the domain $\mathscr{D}$ respectively and V. P. denotes principal value.

If $f(z)$ is regular in $\mathscr{D}$ and on its boundary, then

$$F_i(z_0)=f(z_0).$$

From the authors' summary

1401:

Kung, Sun. On the Schwarz lemma in Einstein spaces of several complex variables. Acta Math. Sinica **7** (1957), 471–476. (Chinese. English summary)

Let D be a simple domain of the complex space $(z^1,\cdots,z^n)$ with Bergman metric $ds^2=dzTd\bar{z}'$. It is assumed that D is an Einstein space with Ricci curvature equal to -1. The following result is proved: if $w^j=f^j(z)$ is a pseudo-conformal mapping of the domain D onto D' and if there exists a positive metric $ds^2=dwH_w d\bar{w}'=dzHd\bar{z}'$ in D' such that its Ricci curvature is less than or equal to -1 and such that $\det H<\det T$ on the boundary ∂D of D, then the following inequality holds in D

$$\left|\frac{\partial(w^1\cdots w^n)}{\partial(z^1\cdots z^n)}\right|^2\leq\frac{\det T}{\det H_w}.$$

In particular, if $\det T\to\infty$ as $z\to\partial D$ and if $w^j=f^j(z)$ is a pseudo-conformal mapping of D onto itself, then

$$\left|\frac{\partial(w^1\cdots w^n)}{\partial(z^1\cdots z^n)}\right|^2\leq\frac{\det T(z)}{\det T(w)}.$$

The results obtained are generalizations of the Schwarz-Cartan lemma, the Schwarz-Pick-Ahlfors lemma and the Schwarz-Bergman lemma in Einstein spaces of several complex variables. *Gu Čao-hao* (RŽ Mat **1959** #4173)

1402:

Grauert, Hans; und Remmert, Reinhold. Bilder und Urbilder analytischer Garben. Ann. of Math. (2) **68** (1958), 393–443.

Ce mémoire contient la démonstration complète des deux théorèmes fondamentaux annoncés en 1957 [C. R. Acad. Sci. Paris **245** (1957), 819–822; MR **19**, 1076]; cf. théorèmes I et II ci-dessous. La démonstration nécessite de longs développements théoriques, dont voici une analyse sommaire.

§ 1. Théorie générale des faisceaux et de la cohomologie. —On développe essentiellement le point de vue de Grothendieck et Godement, attribué ici par erreur à H. Cartan [voir #1583].

§ 2. Espaces annelés; images réciproques et images directes de faisceaux.—Un espace annelé ("geringter Raum") est un espace topologique muni d'un sous-faisceau $\mathfrak{A}$ du faisceau des germes de fonctions continues à valeurs complexes; on suppose que $\mathfrak{A}$ est un faisceau d'anneaux contenant les constantes. On a alors la notion de faisceau de $\mathfrak{A}$-modules, ou $\mathfrak{A}$-faisceau, et une notion de morphismes d'espaces annelés: $(X,\mathfrak{A})\to(Y,\mathfrak{B})$. Pour un tel morphisme $\tau: X\to Y$, et pour tout $\mathfrak{B}$-faisceau $\mathfrak{T}$ sur Y, on définit un $\mathfrak{A}$-faisceau $\mathfrak{S}$ sur X, image réciproque de $\mathfrak{T}$ par τ, noté $\tau^*(\mathfrak{T})$; $\tau^*(\mathfrak{T})$ est un foncteur covariant de $\mathfrak{T}$, additif et exact à droite. Pour tout $\mathfrak{A}$-faisceau $\mathfrak{S}$ sur X, on définit (suivant Leray et Grothendieck) une suite de $\mathfrak{B}$-faisceaux $\tau_0(\mathfrak{S}),\cdots,\tau_q(\mathfrak{S}),\cdots$ sur Y; ce sont des foncteurs covariants additifs; τ_0 est exact à gauche, les τ_q sont les foncteurs dérivés droits de τ_0. Les foncteurs τ^* et τ_0 sont adjoints au sens de Kan, et on a notamment un A-homomorphisme naturel $\tau^*(\tau_0(\mathfrak{S}))\to\mathfrak{S}$. Si cet homomorphisme est surjectif, on dit que $\mathfrak{S}$ est A-simple (relativement à τ); quand Y est réduit à un point, cela exprime que, pour chaque $x\in X$, $\mathfrak{S}_x$ est engendré, comme $\mathfrak{A}_x$-module, par l'ensemble $H^0(X,\mathfrak{S})$ des sections de $\mathfrak{S}$ (on dit alors que $\mathfrak{S}$ est un A-faisceau). On dit que $\mathfrak{S}$ est B-simple (rel. à τ) si les faisceaux $\tau_q(\mathfrak{S})$ sont nuls pour $q\geqq 1$; quand Y est réduit à un point, cela exprime que $H^q(X,\mathfrak{S})=0$ pour $q\geqq 1$ (on dit alors que $\mathfrak{S}$ est un B-faisceau).

§ 3. Faisceaux analytiques cohérents sur les espaces analytiques complexes; diviseurs, fibrés analytiques à fibre vectorielle de dimension 1; énoncé des théorèmes fondamentaux.—Après un rappel de résultats classiques (voir le titre), on arrive à l'essentiel. On sait que tout diviseur sur un espace analytique X définit un fibré analytique de base X, à fibre vectorielle de dimension 1. Soit P^n l'espace projectif complexe de dimension n; soit F le fibré de base P^n défini par n'importe quelle section hyperplane de P^n; notons, avec Serre, $F(k)$ le fibré, produit tensoriel de F par lui-même k fois, et soit $\mathfrak{F}(k)$ le faisceau associé, qui est localement isomorphe au faisceau structural $\mathfrak{O}$ (des germes de fonctions holomorphes). Soient Y un espace analytique quelconque, et p: $Y\times P^n\to P^n$ la projection; on note encore $F(k)$ et $\mathfrak{F}(k)$ les images réciproques p^* de ces fibrés [resp. faisceaux]. Pour tout faisceau analytique cohérent $\mathfrak{S}$ sur $Y\times P^n$, soit $\mathfrak{S}(k)=\mathfrak{S}\otimes\mathfrak{F}(k)$, le produit tensoriel étant pris sur le faisceau structural. On note τ la projection $Y\times P^n\to Y$. Les théorèmes fondamentaux s'énoncent ainsi:

Théorème II_n.—$\mathfrak{S}$ étant un faisceau cohérent sur $Y\times P^n$, les images directes $\tau_q(\mathfrak{S})$ sont des faisceaux cohérents sur Y.

Théorème I_n.—$\mathfrak{S}$ étant un faisceau cohérent sur

$Y \times P^n$, soit Q un ouvert relativement compact de Y; pour k assez grand, la restriction de $\mathfrak{S}(k)$ à $Q \times P^n$ est A-simple et B-simple relativement à $\tau: Q \times P^n \to Q$.

De là on déduit aisément:

Théorème III_n.—Avec les notations du th. I_n, supposons que Q soit holomorphiquement complet; alors la restriction de $\mathfrak{S}(k)$ à $Q \times P^n$ est, pour k assez grand, à la fois un A-faisceau et un B-faisceau. (Si Y est réduit à un point, on retrouve un théorème classique de Serre.)

§§ 4 et 5. Démonstration récurrente des théorèmes I_n et II_n.—Pour des raisons techniques, on introduit l'énoncé $\mathrm{II}_n{}^*$: sous les hypothèses de I_n, la restriction à Q de l'image $\tau_0(\mathfrak{S}(k))$ est un faisceau cohérent pour k assez grand. (Il est clair que $\mathrm{II}_n{}^*$ est une conséquence de II_n.) Au § 4, on démontre I_1 et $\mathrm{II}_1{}^*$. Pour cela, on utilise notamment un résultat intéressant en lui-même (Satz 15): soit $\mathfrak{F}$ un sous-faisceau analytique de $\mathfrak{O}^q$ tel que $H^1(U, \mathfrak{F}) = 0$ pour tout ouvert de Stein U assez petit; alors $\mathfrak{F}$ est cohérent.

Au § 5, on procède par récurrence sur n, en faisant éclater un point de P^n par le procédé de Hopf (qui remplace le point par un P^{n-1}). La récurrence consiste à prouver les implications suivantes:

$$(\mathrm{I}_1^{'}, \mathrm{I}_{n-1}, \mathrm{II}_1{}^0, \mathrm{II}_{n-1}^0) \Rightarrow \mathrm{I}_n$$
$$(\mathrm{II}_1{}^0, \mathrm{II}_{n-1}^0) \Rightarrow \mathrm{II}_n{}^*$$
$$(\mathrm{I}_n, \mathrm{II}_1{}^*, \mathrm{II}_n{}^* \text{ et } \mathrm{II}_{n-1}) \Rightarrow \mathrm{II}_n \quad \text{pour} \quad n \geqq 1;$$

on désigne par $\mathrm{II}_n{}^0$ la partie de l'assertion de II_n qui est relative à l'image $\tau_0(\mathfrak{S})$. *H. Cartan* (Paris)

1403:

Matsushima, Yozô. Fibrés holomorphes sur un tore complexe. Nagoya Math. J. **14** (1959), 1–24.

Let P be a holomorphic principal bundle over a compact complex manifold M with structure group U (a complex Lie group). An automorphism of P is defined to be a complex analytic homeomorphism of the bundle space onto itself commuting with the operation of U. According to Morimoto [same J. **13** (1958), 157–168; MR **20** #2474] the group $F(P)$ of all automorphisms of P has a natural structure of complex Lie group, and there is a natural homomorphism (of complex Lie groups) $F(P) \to A(M)$, where $A(M)$ denotes the group of automorphisms of the compact complex manifold M. P is defined to be homogeneous if the image of $F_0(P)$ (the connected component of the identity in $F(P)$) in $A(M)$ is transitive on M.

The author shows that, if M is a complex torus, P is homogeneous if and only if it has a holomorphic connection. Using this result he is then able to make a general study of bundles over a torus having holomorphic connections. In particular he is able to solve the classification problem completely in the case when $U = \mathrm{GL}(2, C)$. *M. F. Atiyah* (Cambridge, England)

SPECIAL FUNCTIONS

See also 1691, 1845a–b, 1862.

1404:

Mohr, E. Elementarer Beweis für die Produktentwicklung des Sinus und die Partialbruchzerlegung des Cotangens. Z. Angew. Math. Mech. **39** (1959), 78–80.

L'auteur donne une démonstration longue (mais ne faisant intervenir que des connaissances de trigonométrie et d'inégalités faciles) de la formule très classique:

$$\pi \operatorname{cotg} \pi x = \frac{1}{x} + \sum_{\nu=1}^{\infty} \frac{2x}{x^2 - \nu^2}$$

R. Campbell (Caen)

1405:

Wolibner, W. Sur les fonctions dont les intégrales étendues aux surfaces sphériques sont nulles. Colloq. Math. **5** (1957), 66–68.

Let ρ, φ, ϑ denote spherical coordinates, and let the polynomial $V_n(\rho)$, of degree less than n, be even for n even and odd for n odd. For integral values of k, $|k| \leq n$, the integral of the function $\rho^{-2} V_n'(\rho) P_n{}^k(\cos \vartheta) e^{ik\varphi}$ ($P_n{}^k =$ associated Legendre function) over an arbitrary sphere containing the origin is zero. The proof is computational. *P. Henrici* (Los Angeles, Calif.)

1406:

Ragab, F. M. Expansion of an E-function in a series of products of E-functions. Proc. Glasgow Math. Assoc. **3** (1958), 194–195.

This expansion is:

$$\Gamma(\delta-\alpha)\Gamma(\delta-\beta)\Gamma(\alpha+\beta-\delta)[\Gamma(\alpha)\Gamma(\beta)]^{-1}E(\alpha, \beta, \gamma : \delta : z) = \sum_{r=0}^{\infty} \frac{z^{-2r}}{r!\Gamma(\gamma+r)} E(\gamma+r, \alpha+\beta-\delta+r :: z) \times E\left(\begin{matrix}\gamma+r, \delta-\alpha+r, \delta-\beta+r : z \\ \delta+r\end{matrix}\right),$$

where $|\operatorname{amp} z| < \pi$, $z \neq 0$, $R(\alpha+\beta) > R(\delta) > R(\alpha) > 0$, $R(\delta-\beta) > 0$. In proving this expansion the author uses the two following formulae:

$$E(p; \alpha_n : q; \rho_s : z) = \Gamma(\alpha_p)\left[\prod_{n=1}^{q} \Gamma(\rho_n - \alpha_n)\right]^{-1} \times \prod_{n=1}^{q} \int_0^1 \lambda_n{}^{\alpha_n - 1}(1-\lambda_n)^{\rho_n - \alpha_n - 1} d\lambda_n \prod_{n=q+1}^{p-2} \int_0^\infty e^{-\lambda_n} \lambda_n{}^{\alpha_n - 1} d\lambda_n \times \int_0^\infty e^{-\lambda_{p-1}} \lambda_{p-1}^{\alpha_{p-1}-1} (1 + \lambda_1 \lambda_2 \cdots \lambda_{p-1}/z)^{-\alpha_p} d\lambda_{p-1}$$

[T. M. MacRobert, *Functions of a complex variable*, 4th ed., Macmillan, London 1954; p. 352], and

$$\int_0^\infty e^{-\lambda} \lambda^{\delta-\alpha-\beta-1}(\lambda+z)^{-\gamma} E(\alpha, \beta, \gamma : \delta : \lambda+z) d\lambda = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\delta-\alpha-\beta)}{\Gamma(\delta-\alpha)\cdot\Gamma(\delta-\beta)} z^{-\gamma} E(\delta-\alpha, \delta-\beta, \gamma : \delta : z)$$

$$(R(\delta-\alpha-\beta) > 0, \ |\operatorname{amp} z| < \pi, \ z \neq 0).$$

[C. B. Rathie, Proc. Glasgow Math. Assoc. **2** (1955), 170–172; MR **17**, 846.] *S. C. van Veen* (Delft)

1407:

Toscano, Letterio. Polinomi ortogonali o reciproci di ortogonali nella classe di Appell. Matematiche, Catania **11** (1956), 168–174 (1957).

A sequence of polynomials $f_0(x)$, $f_1(x)$, $\cdots$ (subscript = degree) is called an Appell sequence if $f_n'(x) = n f_{n-1}(x)$

($n=1, 2, \cdots$). Results: (1) Every Appell sequence consisting of orthogonal polynomials (with respect to a suitable weight function) can be represented in terms of Hermite polynomials. (2) Every Appell sequence for which the reciprocal polynomials $x^n f_n(x^{-1})$ are orthogonal can be represented in terms of certain Laguerre polynomials. *P. Henrici* (Los Angeles, Calif.)

1408:

Rajagopal, A. K. On Bessel polynomials. Boll. Un. Mat. Ital. (3) **13** (1958), 418–422. (Italian summary)

The so-called Bessel polynomials [see W. A. Al-Salam, Duke Math. J. **24** (1957), 529–545; MR **19**, 849] after suitable modification satisfy Truesdell's F-equation [see C. Truesdell, *An essay toward a unified theory of special functions*, Princeton Univ. Press, Princeton, N.J., 1948; MR **9**, 431]. A number of trivial consequences of this fact are presented in the paper under review.
P. Henrici (Los Angeles, Calif.)

1409:

Lakshmana Rao, S. K. Turan's inequality for the general Laguerre and Hermite functions. Math. Student **26** (1958), 1–6.

Turan's inequality

$$\Delta_n(x) = [H_n(x)]^2 - H_{n+1}(x)H_{n-1}(x) \geqq 0$$

is established for the Hermite functions $H_n(x)$, valid for real x and $n \geqq 0$. It is further shown that the conditions (i) $H_n(x)$ is a polynomial of degree n ($n=0, 1, 2, \cdots$); (ii) $H_0(x)=1$, $H_1(x)=2x$; and (iii) $d[\exp(-x^2)\Delta_n(x)]/dx = -2\exp(-x^2)H_{n-1}(x)H_n(x)$, characterize the Hermite polynomials. Turan's inequality is also established for the Laguerre functions $L_n^{(\alpha)}(x)$ when $x>0$, $n>0$, $\alpha \geqq 0$, and conditions are found which characterize the Laguerre polynomials. *C. A. Swanson* (Vancouver, B.C.)

1410:

Brafman, Fred. An ultraspherical generating function. Pacific J. Math. **7** (1957), 1319–1323.

Let $\rho=(1-2vt+t^2)^{1/2}$, $y=tu\rho^{-1}$, $w=2(v-t)\rho$, $r=(1-2yw+y^2)^{1/2}$. The function

$$\rho^{-2\alpha-1}P_k^{(\alpha,\alpha)}(r+y)P_k^{(\alpha,\alpha)}(r-y)$$

is expanded in a series of the functions $P_n^{(\alpha,\alpha)}(v)t^n$ ($n=0, 1, 2, \cdots$) with coefficients depending on u. The result generalizes a formula due to S. O. Rice [Duke Math. J. **6** (1940), 108–119; MR **1**, 234].
P. Henrici (Los Angeles, Calif.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 1716, 1749.

1411:

Babkin, B. N. The theorem of S. A. Čaplygin on differential inequalities. Mat. Sb. N.S. **46** (88) (1958), 389–398. (Russian)

The author obtains theorems giving certain bounds on the solutions of linear systems of ordinary differential equations by extensions of a theorem of Čaplygin [*Sobranie sočinenii*, tom I. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1948; MR **14**, 609] on differential inequalities. The proofs in the paper are by straightforward techniques of analysis.

In (1) $L[y]=y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_n(x)y=f(x)$; $y^{(k)}(x_0)=y_0^{(k)}$ ($k=0, 1, \cdots, n-1$), let the coefficients $p_i(x)$ be continuous on $[x_0, x_1]$. Let $t(x)\in C^n[x_0, x_1]$ satisfy the initial conditions of (1) and also the differential inequality (2) $L[t]-f(x)<0$. Let $v(x)\in C^n[x_0, x_1]$ satisfy the initial conditions of (1) and the differential inequality

$$(3)\qquad L[v]-f(x) > \textstyle\sum_{k=2}^n (M_k+|M_k|)(v-t)^{(n-k)},$$

where $M_k=\max p_k(x)$ on $[x_0, x_1]$. Then the solution $y(x)$ of (1) satisfies $y^{(k)}(x)<v^{(k)}(x)$ ($k=0, 1, \cdots, n-1$) for $x_0<x\leqq x_1$.

This result can be extended to give bounds on the difference between $y(x)$ and $t(x)$. Let $\varepsilon_1=\max|L[t]-f(x)|$, $m=\min p_1(x)$, $N_k=\max p_k(x)$, all for $x_0\leqq x\leqq x_1$, and let $z_1(x)$ be that solution of (4) $z_1^{(n)}+mz_1^{(n-1)}-\sum_{k=2}^n N_k z_1^{(n-k)}=\varepsilon_1$ for which $z_1^{(k)}(x_0)=0$ ($k=0, 1, \cdots, (n-1)$). Then

$$|y^{(k)}(x)-t^{(k)}(x)|<z_1^{(k)}(x)\quad\text{for}\quad x_0<x\leqq x_1$$
$$(k=0, 1, 2, \cdots, (n-1)).$$

The analogous result is obtained for a system $y_i'+\sum_{k=1}^n p_{ik}(x)y_k=f_i(x)$, ($i=1, 2, \cdots, n$). The results are illustrated with examples. The results are extended to a nonlinear system $y_i'=f_i(x, y_1, \cdots, y_n)$ ($i=1, 2, \cdots, n$) assuming continuity of the f_i and their partial derivatives with respect to the y_k. *W. S. Loud* (Minneapolis, Minn.)

1412:

Łojasiewicz, S. Sur les équations du mouvement d'un système holonome. Ann. Polon. Math. **5** (1958/59), 247–256.

Consider a system of differential equations of second order

$$(1)\qquad \frac{d^2x}{dt^2}=f(t, x, \dot{x})-M^2F(t, x, \dot{x}),$$

where x, f, F are n-vectors and M a large parameter. It is assumed that the force term F vanishes on an $n-p$ dimensional surface S: $\varphi_\nu(x, t)=0$ ($\nu=1, \cdots, p$). Moreover, the matrix F_x is assumed to be symmetric and positive definite in S. The solutions of (1) for large M are compared with the solutions of

$$(2)\qquad \frac{d^2y}{dt^2}=f(t, y, \dot{y})-\sum_{\nu=1}^p \lambda_\nu\varphi_{\nu x};\quad \varphi_\nu(t, y)=0.$$

Under some smoothness conditions it is shown that the solutions of (1) and (2) satisfy

$$|x-y|+|\dot{x}-\dot{y}|\leqq C/M;\quad |\varphi_\nu(t, x)|<C/M^2$$

for a finite t-interval, provided they satisfy such inequalities (with another C) initially. [See also H. Rubin and P. Ungar, Comm. Pure Appl. Math. **10** (1957), 65–87; MR **19**, 477.] *J. Moser* (Cambridge, Mass.)

1413:

Maurin, L. N. Solution of linear differential equations by the method of the total differential. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 2, 3–8. (Russian)

The method given in the paper is a generalization of the Laplace method. The author assumes the solution $y(x)$ of the linear equation $L(y)=0$ in the form

$$(1)\qquad y(x) = \int_A^B \alpha(k) \exp(F(k, x))dk$$

and determines the functions $\alpha(k)$ and $F(k, x)$ from the identity

$$\alpha(k)\Lambda\left(\frac{\partial F}{\partial x}\right) \exp(F(k, x)) = \frac{\partial}{\partial k}[\alpha(k) \exp(F(k, x))],$$

where Λ is a nonlinear operator of order one less than that of L, which we obtain by applying L under the integral sign in (1). The method is used to the solution of the Weber and Mathieu equation. *M. Zlámal* (Brno)

1414:

Bandić, Ivan. Sur une classe d'équations différentielles indéterminées du deuxième ordre. C. R. Acad. Sci. Paris, **247** (1958), 800–803.

It is shown that the solutions $\{y(x), z(x)\}$ of the indeterminate equation

$$\frac{y''}{y}+a_0(x)\frac{z''}{z}+a_1(x)\frac{y'}{y}+a_2(x)\frac{z'}{z} = a_3(x) \quad (a_0 \neq -1)$$

are given by

$$y = c_1 w \exp\left\{\frac{1}{2}\int\frac{2a_0\lambda-a_1-a_2}{1+a_0}dx\right\},$$

$$z = c_2 w \exp\left\{\frac{1}{2}\int\frac{a_1+a_2+2\lambda}{1+a_0}dx\right\},$$

where c_1 and c_2 are arbitrary constants, w is an arbitrary function, and λ is a function of w. Applications to special cases are given. *J. Elliott* (New York, N.Y.)

1415:

Strelic, Š. I. Growth of solutions of linear differential equations of second order. Mat. Sb. N.S. **46** (88) (1958), 433–450. (Russian)

The author considers the rate of growth of solutions of the equation

$$(1)\qquad w''+P(z)w'+Q(z)w = 0$$

in the neighborhood of singular points. Here P and Q are rational functions. Let z_0 be a pole of P or Q. Then in a neighborhood of z_0 the solutions of (1) have one of the forms:

$$(a)\qquad w(z) = c_1(z-z_0)^\alpha f_1(z)+c_2(z-z_0)^\beta f_2(z),$$

$$(b)\qquad w(z) = (z-z_0)^\alpha[c_1(f_1(z)+f_2(z)\log(z-z_0))+c_2f_2(z)],$$

where α, β are complex numbers and $f_1(z)$, $f_2(z)$ are single-valued analytic functions in a deleted neighborhood $0<|z-z_0|<d$ of z_0. The author considers only case (a), and assumes that f_1 and f_2 have essential singularities at z_0. He defines a notion of order and type for f_1 and f_2 as $z\to z_0$ (in analogy to the theory of entire functions) and determines the order and type of f_1, f_2 in terms of properties of P and Q. He treats fully the case when $z_0=\infty$. He also handles some other problems—for example, when does there exist a solution of (1) that is real on a segment of the real axis? *A. Shields* (Ann Arbor, Mich.)

1416:

Shimanov, S. N. Finding the characteristic exponents of systems of linear differential equations with periodic coefficients. J. Appl. Math. Mech. **22** (1958), 526–531 (382–385 Prikl. Mat. Meh.).

This is an extension of the results obtained in one of the author's earlier papers [Prikl. Mat. Meh. **16** (1952), 129–146; MR **13**, 745]. Two theorems are proved on the existence and analyticity of the characteristic exponents of the considered system of differential equations.
H. P. Thielman (Ames, Iowa)

1417:

Hale, J. K. A short proof of a boundedness theorem for linear differential systems with periodic coefficients. Arch. Rational Mech. Anal. **2** (1958/59), 429–434.

The principal result of the paper is a boundedness theorem for a system of second order, homogeneous, linear differential equations with periodic coefficients and a small parameter. Under certain conditions of symmetry all solutions are bounded for sufficiently small values of the parameter. Results of this type have been obtained by Cesari, Hale, and Golomb, among others, using methods of successive approximation. Here a general boundedness theorem is established by an elementary argument that shows under the conditions placed on the system that the characteristic exponents are all pure imaginary. *J. P. LaSalle* (Baltimore, Md.)

1418:

Wintner, Aurel. On the "enveloping" properties of the Maclaurin series of cos t, sin t and e^{-t}. Simon Stevin **31** (1957), 149–155.

L'auteur lie le fait que les sommes partielles des séries de Maclaurin des fonctions $\cos t$, $\sin t$, e^{-t} tendent vers ces fonctions d'une manière qu'on peut traiter de "pincer-like" (expression de Weyl), aux faits plus généraux concernant les solutions d'équations différentielles de la forme $x''+f(t)x=0$ $(x=x(t))$. *S. Mandelbrojt* (Paris)

1419:

Šeda, V. On the transformation of the integrals of ordinary linear differential equations of the second order in the complex domain. Acta Fac. Nat. Univ. Comenian. Math. **2** (1958), 229–254. (Slovak. Russian and English summaries)

Borůvka [Ann. Mat. Pura Appl. (4) **41** (1956), 325–342; MR **20** #1814] dealt with the transformation of the integrals of the linear equation of the second order in the real domain. The author studies this problem in the complex domain. The main result is the following theorem. Let $Q(x)$ and $q(x)$ be holomorphic functions and let $\{z, x\}$ denote the Schwarz derivative,

$$\{z, x\} = \frac{1}{2}\frac{z'''}{z'}-\frac{3}{4}\left(\frac{z''}{z'}\right)^2.$$

Then the solution $z(x)$ of the equation

$$-\{z, x\}+Q(z)z'^2 = q(x)$$

satisfying the initial conditions $z(x_0)=z_0$, $z'(x_0)=z_0'$, $z''(x_0)=z_0''$ is a composite analytic function $f^{-1}g$. Here f^{-1} is the inverse function of the solution $f(z)$ of the equation

$-\{f, z\}=Q(z)$ determined by the initial conditions $f(z_0)=f_0, f'(z_0)=f_0', f''(z_0)=f_0''$, where $f_0, f_0'(\neq 0)$, f_0'' are arbitrary values, and g is the solution of $-\{g, x\}=g(x)$ determined by $g(x_0)=f_0$, $g'(x_0)=f_0'z_0'$, $g''(x_0)=f_0''z_0'^2+f_0'z_0''$.

M. Zlámal (Brno)

1420:

Zadiraka, K. V. The construction of upper and lower estimates for eigenvalues of boundary problems. Bul. Inst. Politehn. Iaşi (N.S.) 4 (8) (1958), 3–16. (Russian. English and Romanian summaries)

This is a study of a method for the determination of upper and lower estimates of the simple eigenvalues of the selfconjugate boundary value problem

$$(1)\quad (-1)^k\frac{d^{2k}y}{dx^{2k}}+[q(x)y-\lambda p(x)]y = 0 \quad (0 \leqq x \leqq l),$$

$$(2)\quad \sin a_s y^{(s)}(0)+\cos a_s y^{(2k-1-s)}(0) = 0 \quad (s = 0, 1, \cdots, k-1),$$

$$(3)\quad \sin b_s y^{(s)}(l)+\cos b_s y^{(2k-1-s)}(l) = 0 \quad (s = 0, 1, \cdots, k-1),$$

where $p(x)$, $q(x)$ are real continuous functions in $[0, l]$, and a_s, b_s real constants. For every simple eigenvalue $\lambda^{(j)}$ the method gives rise to two sequences of estimates

$$\underline{\lambda}_n^{(j)} \leqq \underline{\lambda}_{n+1}^{(j)} \leqq \cdots \leqq \lambda^{(j)} \leqq \cdots \leqq \bar{\lambda}_{n+1}^{(j)} \leqq \bar{\lambda}_n^{(j)},$$

n sufficiently large, both convergent toward $\lambda^{(j)}$ as $n\to+\infty$.—First the author discusses the Cauchy problem (4) $(d^{2k}y/dx^{2k})+Q(x)y=0$, (5) $y(0)=y_0, \cdots, y^{(2k-1)}(0)=y_{2k-1}$. By replacing (4) by $(d^{2k}y/dx^{2k})+Cy=(C-Q(x))y$, where $C=\pm M^{2k}$, or $C=\pm m^{2k}$, and $M^{2k}=|\max Q(x)|$, $m^{2k}=|\min Q(x)|$, $0\leqq x\leqq l$, and by considering the usual successive approximations of the equivalent Volterra integral equation, the author obtains minorants and majorants $\underline{y}_n(x)$, $\bar{y}_n(x)$ of the solutions of (4), (5), with $\underline{y}_n(x)\leqq \underline{y}_{n+1}(x)\leqq \cdots \leqq y(x)\leqq \cdots \leqq \bar{y}_{n+1}(x)\leqq \bar{y}_n(x)$, all approaching $y(x)$ as $n\to+\infty$, $0\leqq x\leqq l$ [R. Bellman, *Stability theory of differential equations*, McGraw-Hill, New York-Toronto-London, 1953; MR 15, 794; S. A. Čaplygin, *Sobranie sočinenii*, *Tom I*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1948; MR 14, 609].—Then the problem (1), (2), (3) is attacked. First the minorants and majorants $\underline{y}_{in}(x, \lambda)$, $\bar{y}_{in}(x, \lambda)$ are determined of the k independent solutions $y_i(x, \lambda)$ of (1) with initial conditions $y^{(k-i)}(0)=-\cos a_{k-i}$, $y^{(k+i)}(0)=\sin a_{k-i}$, $y^{(h)}(0)=0$, $h=0, 1, \cdots, 2k-1$, $h\neq k\pm i$, $(i=1, 2, \cdots, k)$. These k functions $y_i(x, \lambda)$ satisfy the k initial conditions (2) identically. Then the determinant $D(\lambda)$, which is obtained by replacing $y=C_1y_1+\cdots+C_ny_n$ in the k conditions (3), gives rise to expressions $\underline{D}_n(\lambda)$ $[\bar{D}_n(\lambda)]$ by simply replacing each $y_i(x, \lambda)$ by either $\underline{y}_{in}(x, \lambda)$, or $\bar{y}_{in}(x, \lambda)$, according as the corresponding coefficient is negative or positive [positive or negative], and analogously for the derivatives. The inequalities $\underline{D}_n(\lambda)\leqq D(\lambda)\leqq \bar{D}_n(\lambda)$ hold. Finally, the zeros $\underline{\lambda}_n^{(j)}$, $\bar{\lambda}_n^{(j)}$ of $\underline{D}_n^{(\lambda)}$, $\bar{D}_n^{(\lambda)}$ are the sought for lower and upper estimates.—The cases $k=1$ (Sturm-Liouville) and $k=2$ are studied in more detail. Numerical examples show the remarkable strength of the approximations.

L. Cesari (Baltimore, Md.)

1421:

Bukovics, Erich. Lineare Eigenwertaufgaben in der Mechanik (Ein Beitrag zur Theorie der natürlichen Eigenwertprobleme). Acta Phys. Austriaca 12 (1958/59), 262–303.

Die Theorie der Eigenwertaufgaben bei gewöhnlichen Differentialgleichungen ist um 1940 von E. Kamke sehr gefördert worden, indem er die Ergebnisse der Sturm-Liouville'schen Theorie auf Differentialgleichungen höherer Ordnung und allgemeine Randbedingungen ausgedehnt hat. Seine Betrachtungen umfassten jedoch noch nicht den Fall, wo der Eigenwert λ nicht nur in der Differentialgleichung sondern auch in den Randbedingungen auftritt. Schon relativ einfache Schwingungs- und Stabilitätsprobleme der Mechanik zeigen diese Erscheinung. Um ihr Rechnung zu tragen, formulierten E. Stiefel und H. Ziegler ausgehend von der Variationsrechnung ein Eigenwertproblem in der folgenden Weise. Gegeben seien in einem Grundintervall (a, b) zwei bilineare Funktionale $\phi(\zeta, \eta)$ und $\psi(\zeta, \eta)$. Dabei sind ζ, η zwei Funktionen im Grundintervall. ϕ und ψ setzen sich zusammen einerseits aus Integralen über das Grundintervall enthaltend ζ, η und die Ableitungen dieser Funktionen, und andererseits aus Randteilen, in welchen die Randwerte von ζ, η und die Randwerte der Ableitungen vorkommen. Ausserdem sind wesentliche Randbedingungen gegeben, welche die Randwerte von ζ, η und einiger Ableitungen dieser Funktionen linear und homogen verknüpfen. Eine Funktion heisse zulässig, wenn sie diese wesentlichen Randbedingungen erfüllt. Ein natürliches Eigenwertproblem nach Stiefel und Ziegler besteht darin, eine Funktion ζ und einen Eigenwert λ so zu finden, dass für jede zulässige Funktion η gilt $\phi(\zeta, \eta)-\lambda\psi(\zeta, \eta)=0$. Führt man diese Variation aus, so erhält man eine Differentialgleichung für ζ und neue Randbedingungen, die zu den wesentlichen hinzutreten und restliche Randbedingungen heissen. Diese restlichen Randbedingungen können den Eigenwert enthalten, während die wesentlichen mit ihm ja noch gar nichts zu tun haben.

N. J. Lehmann hat mit Hilfe der Integralgleichungstheorie Eigenwertprobleme behandelt, die weitgehend von dieser Allgemeinheit sind. Die vorliegende Arbeit enthält nun eine erschöpfende Theorie der natürlichen Eigenwertprobleme ohne Benutzung von Integralgleichungen. Zunächst werden die Definitionen und die Transformation auf eine Normalform erneut dargestellt. Dann folgt eine eingehende Diskussion der restlichen Randbedingungen, die mit Hilfe der Matrizenrechnung explizit aufgestellt werden. Durch geeignete Transformationen wird eine "Hauptform" für diese restlichen Randbedingungen hergestellt. Besonders interessant ist die darauf folgende Lösung des folgenden Umkehrproblems. Gegeben sei ein Eigenwertproblem in Differentialgleichungsform mit wesentlichen und restlichen Randbedingungen, wobei die letzteren den Eigenwert linear enthalten dürfen. Unter welchen Bedingungen ist dies ein natürliches Eigenwertproblem? Die notwendigen und hinreichenden Bedingungen werden hergeleitet. Sie haben die Gestalt linearer Relationen zwischen den Koeffizienten der restlichen Randbedingungen, wobei eine gewisse Matrix symmetrisch sein muss, was natürlich eine Art Selbstadjungiertheit bedeutet. Die Umrechnungen werden durch ein übersichtliches Zahlenbeispiel illustriert.

In zweiten Teil seiner Arbeit befasst sich der Verfasser mit definiten und semidefiniten natürlichen Eigenwertproblemen. Nach Abklärung dieser Begriffe werden die Sätze über die Abzählbarkeit und Realität der Eigenwerte sowie über die Orthogonalität der Eigenfunktionen

hergeleitet, die man auf Grund der bekannten Theorie einfacherer Eigenwertprobleme erwartet. Aber auch die schwierigeren Sätze über die Minimaleigenschaften der Eigenwerte und die Vollständigkeit des Systems der Eigenfunktionen werden bewiesen unter Benutzung der Kamke'schen Beweistechnik, die sich auf den vorliegenden Problemkreis übertragen lässt.

In einem Anhang gibt der Verfasser noch Rechenvorschriften für die Transformation eines natürlichen Eigenwertproblems auf die Normalform. Ueberhaupt ist die Darstellung seiner Methoden so explizit gestaltet, dass diese Methoden zur Lösung von Problemen der angewandten Mathematik herangezogen werden können.

E. Stiefel (Zürich)

1422:

Bellman, Richard; and Fort, Tomlinson. On convergent perturbation expansions. Quart. Appl. Math. **17** (1959), 96–98.

Let $\Lambda=\Lambda(\varepsilon)$ denote the smallest characteristic value and u a corresponding characteristic function of the boundary value problem

$$u''+\lambda(f(x)+\varepsilon g(x))u = 0, \quad u(0) = u(1) = 0,$$

where $f(x)$ and $g(x)$ are continuous on $0\leq x\leq 1$ and satisfy $f(x), g(x)\geq a^2>0$ for $0\leq x\leq 1$. Let λ_0 and u_0 be respectively the first characteristic value and characteristic function of the boundary value problem

$$u''+\lambda f(x)u = 0, \quad u(0) = u(1) = 0,$$

where $u_0(x)$ is normalized by $u_0'(0)=1$. Expansions of the type

$$u = \sum_0^\infty u_n(h(\varepsilon))^n \quad \text{and} \quad \Lambda = \sum_0^\infty \lambda_n(h(\varepsilon))^n,$$

convergent for all $\varepsilon\geq 0$ and reducing to u_0 and λ_0 when $\varepsilon=0$, are obtained. *C. R. Putnam* (Lafayette, Ind.)

1423:

Ráb, Miloš. Über lineare Perturbationen eines Systems von linearen Differentialgleichungen. Czechoslovak Math. J. **8 (83)** (1958), 222–229. (Russian summary)

Let $A(t)$ and $B(t)$ be two n-by-n matrices continuous for $t\geq t_0$ and $Y(t)$ a fundamental matrix for the equation $y'=A(t)y$. The author proves that under the condition

$$\int_{t_0}^\infty \| Y^{-1}(t)B(t)Y(t)\| dt<\infty$$

the solutions $z(t)$ of $z'=[A(t)+B(t)]z$ are of the form $z(t)=Y(t)[c+o(1)]$, where c is a constant vector, and uses this result to derive some asymptotic formulae for the solutions of the second and third order linear equations.

M. Zlámal (Brno)

1424:

Sibuya, Yasutaka. Non-linear ordinary differential equations with periodic coefficients. Funkcial. Ekvac. **1** (1958), 121–204.

The paper is a study of the solutions of a system of nonlinear differential equations $dx/dt=f(t,x)$ (in vector notation) in the vicinity of the equilibrium solution $x=0$. The vector function f is assumed representable by a power series in $x_1, \cdots, x_n$, with coefficients depending on t. A variety of results on normal forms for the equations are obtained and applied to discuss the nature of the solutions, in particular when f is periodic in t. For the most part the results are contained in earlier papers of the author, M. Hukuhara, M. Urabe, E. A. Coddington and N. Levinson [see, for example, Y. Sibuya, J. Fac. Sci. Tokyo Sect. I. 7 (1954), 19–32, 107–127, 229–241, 243–254; MR **15**, 872; **16**, 1026]. *W. Kaplan* (Ann Arbor, Mich.)

1425:

Akutowicz, E. J. The ergodic property of the characteristics on a torus. Quart. J. Math. Oxford Ser. (2) **9** (1958), 275–281.

Let $X(x,y)$ and $Y(x,y)$ be real valued functions having period 1 in both of their (real) arguments, so that X and Y may be considered to be functions on a torus; they are supposed to have continuous second derivatives. Moreover it is assumed that $X^2+Y^2\neq 0$ for all x, y. It is proved that then there exists a unique normalized measure on the torus such that the flow defined by the differential equations

$$\frac{dx}{dt} = X(x,y), \quad \frac{dy}{dt} = Y(x,y)$$

is ergodic with respect to this measure provided the rotation number is irrational. (For the definition of the rotation number see, e.g., Coddington and Levinson's book *Theory of ordinary differential equations* [McGraw-Hill, New York-Toronto-London, 1955; MR **16**, 1022], pp. 406 and 408.) Concerning the history of the problem and results by previous authors used in the proof of the above theorem we refer to the introduction of the paper. We draw, however, attention to the fact that the existence of an invariant measure is not assumed but proved.

E. H. Rothe (Ann Arbor, Mich.)

1426:

Volosov, V. M. Solutions of some perturbed systems in the neighborhood of periodic motions. Dokl. Akad. Nauk SSSR **123** (1958), 587–590. (Russian)

1427:

Minc, R. M. Limit cycle in three-dimensional space with one characteristic number distinct from zero. Dokl. Akad. Nauk SSSR **125** (1959), 38–41. (Russian)

The system under discussion is the analytic system

$$\dot{x} = P(x,y,z), \quad \dot{y} = Q(\cdots), \quad \dot{z} = R(\cdots),$$

with a periodic solution of which only one of the two usual non-zero characteristic numbers is not zero. Let U, V be coordinates in a normal plane Π to the limit cycle γ with $U=V=0$ at the point where γ meets Π. The usual transformation from a point (U_0, V_0) of Π to its consequent may be given the form

$$U = \lambda_1 U_0+F_2(U_0, V_0), \quad V = V_0+\Phi_2(U_0, V_0),$$

where λ_1 is the characteristic exponent $\neq 1$ and F_2, Φ_2 begin with terms of degree at least two. Let $U_0=F(V_0)$ be the solution of $(\lambda_1-1)U_0+F_2(U_0, V_0) = 0$. Then $\Phi_2(F(V_0), V_0)=\Delta_m V_0^m(1+\theta(V_0))$, where $\theta(0)=0$. The behavior of the paths near γ is then dictated by m, the sign of Δ_m and of $G=\oint_\gamma \operatorname{grad}(P,Q,R)dt$.

One must also take into account the two integral surfaces S_1, S_2 through γ. Then there are the following cases. (A_1) $G<0$, m odd, $\Delta_m<0$. γ is stable. (A_2) The same with $\Delta_m>0$. Say on S_1 all paths $\to\gamma$ as $t\to+\infty$, and on S_2 as $t\to-\infty$. The other paths remain at a finite distance from γ (saddle-like behavior). (A_3) $G<0$, m even. On one side, say, of S_1 and on S_1 all paths $\to\gamma$ as $t\to+\infty$. On the other side the paths on $S_2\to\gamma$ as $t\to-\infty$. The other paths remain at a finite distance from γ.

(B_1) $G>0$, m odd, $\Delta_m<0$. Like (A_2). (B_2) The same with $\Delta_m>0$. Like (A_1) if the sign of t is reversed (unstable limit-cycle). (B_3) $G>0$, m even. Like (A_3) if the sign of t is reversed.

[References: Andronov and Leontovič, Mat. Sb. (N.S.) **40** (**82**) (1956), 179–224; MR **19**, 36; Lattès, Ann. Mat. Pura Appl. (3) **13** (1906), 1–138.]

S. Lefschetz (Mexico, D.F.)

1428:

Anan'eva, G. V.; and Balaganskiĭ, V. I. Oscillation of the solutions of certain differential equations of high order. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 135–140. (Russian)

A classical result of Kneser for the equation (1) $y^{(n)}+f(x)y=0$, $f(x)$ continuous, with (2) $\lim f(x)>0$ as $x\to+\infty$, states that for n even all non-trivial solutions of (1) are oscillatory for $x\to\infty$, and for n odd every solution which is not oscillatory has the property that $y^{(k)}(x)\to 0$ as $x\to\infty$ $(k=0, 1, \cdots, n-1)$. The authors prove first the same alternative statement where condition (2) is replaced by $f(x)>0$, $\int^\infty x^{n-2}f(x)dx=\infty$. An example is given showing that the condition $\int^\infty x^{n-1}f(x)dx=\infty$ does not suffice. The authors prove then an analogous extension of the previous statement for the nonlinear equation $Y_n+f(x,y)=0$, where

$$Y_n=\frac{d}{dx}\left\{g_{n-1}\frac{d}{dx}\left[\cdots\frac{d}{dx}\left(g_1\frac{dy}{dx}\right)\cdots\right]\right\},$$

and where $\operatorname{sgn} f(x,y)=\operatorname{sgn} y$ for all $x\geqq x_0$, $|f(x,y)|/\varphi(x)\to\infty$ as $|y|\to\infty$ uniformly with respect to x, with $\varphi(x)>0$, $g_k(x)>0$, $\int^\infty\varphi(x)dx=\infty$, $\int^\infty g_k^{-1}(x)dx=\infty$ $(k=1, \cdots, n-1)$.

L. Cesari (Baltimore, Md.)

1429:

Leighton, Walter; and Nehari, Zeev. On the oscillation of solutions of self-adjoint linear differential equations of the fourth order. Trans. Amer. Math. Soc. **89** (1958), 325–377.

This paper gives a comprehensive account of the zeros of solutions of

$$(1_\pm)\qquad (r(x)y'')''\pm p(x)y=0,$$

where $r>0$, $p>0$ and $r(x)\in C^2$, $p(x)\in C^0$ for $x\geqq 0$. Although certain regular patterns of behavior are found, the complete results are difficult to describe in a brief review. A few will be indicated below.

A function $y(x)\not\equiv 0$ is called non-oscillatory or oscillatory according as it does not or does have an infinity of different zeros. A differential equation is called non-oscillatory if all of its solutions are non-oscillatory. It is shown that one of the differences between ($1_\pm$) is the fact that either all or no solutions of (1_+) are non-oscillatory, while some (but not necessarily all) solutions of (1_-) are non-oscillatory.

The first part of the paper concerns (1_-). Let $a\geqq 0$. If some solution $y=y(x)\not\equiv 0$ of (1_-) has at least $n+3$ zeros $a=a_1\leqq a_2\leqq\cdots\leqq a_{n+3}$, the number $\eta_n(a)=\min a_{n+3}$ for all such solutions is called the nth conjugate point of a. Equation (1_-) is oscillatory if and only if every $a>0$ has infinitely many conjugate points $\eta_1(a)$, $\eta_2(a)$, $\cdots$. If $y=y(x)$ is a solution of (1_-) satisfying $y(a)=y'(a)=0$, then the (finite or infinitely many) zeros of $y(x)$ in (a,∞) are separated by the points $x=\eta_n(a)$. A comparison theorem of the following type is obtained: Let $(r_1(x), p_1(x))$ satisfy the conditions imposed on $(r(x), p(x))$, let $r\geqq r_1>0$, $0<p\leqq p_1$ and $(r_1y'')''-p_1y=0$ be non-oscillatory, then (1_-) is non-oscillatory. Some conditions for (1_-) to be oscillatory or non-oscillatory are discussed; e.g., if $\limsup x^{-2-\alpha}r(x)<1$, $\liminf 16x^{2-\alpha}p(x)>(1-\alpha^2)^2$ as $x\to\infty$ for some constant α, then (1_-) is oscillatory, while if $\liminf x^{-2-\alpha}r(x)>1$, $\limsup 16x^{2-\alpha}p(x)<(1-\alpha^2)^2$, then ($1_-$) is non-oscillatory. Another sufficient condition for (1_-) to be non-oscillatory is that $\int^\infty r^{-1}(x)(\int_x^\infty t^2p(t)dt)dx<\infty$. Some of the proofs in this part of the paper involve a min-max principle and the eigenvalue problem $(ry'')''-\lambda py=0$, $y(a)=y'(a)=y(b)=y'(b)=0$.

The second part of the paper concerns (1_+). For (1_+), the conjugate points $\eta_1(a)$, $\eta_2(a)$, $\cdots$ of $x=a$ are defined as the successive zeros (if any) on (a,∞) of the "principal" solution $y=y(x,a)$ of (1_+) determined by the initial conditions $y(a)=y'(a)=y''(a)=0$ and normalization $r(a)y'''(a)=1$. On its interval of existence, $\eta_n(a)$ is a continuous, strictly increasing function of a. If $\eta_n(a)$ exists and $y=y(x)$ is a solution of ($1_\pm$) vanishing at $x=a$, then $y(x)$ has at least n zeros on $[a, \eta_n(a)]$ and at most as many on $[a, \eta_n(a)]$ as the principal solution $y(x,a)$. Let r, p, r_1, p_1 be as above, $r\geqq r_1>0$, $0<p\leqq p_1$, and $(r_1y'')''+p_1y=0$ be non-oscillatory, then (1_+) is non-oscillatory. This implies, e.g., that if $\limsup x^{-2-\alpha}r(x)<1$, $\liminf 4x^{2-\alpha}p(x)>\alpha^2$ as $x\to\infty$ for some constant α, then (1_+) is oscillatory, while if $\liminf x^{-2-\alpha}r(x)>1$, $\limsup 4x^{2-\alpha}p(x)<\alpha^2$, then ($1_+$) is non-oscillatory. In the second part of the paper, the auxiliary function $\varphi(x)=ry'y''-y(ry'')'$ belonging to a solution $y=y(x)$ plays an important role.

P. Hartman (Baltimore, Md.)

1430a:

Chin, Yuan-shun. Über die algebraischen Grenzzyklen zweiten Grades der Differentialgleichung

$$\frac{dy}{dx}=\frac{\sum_{0\leq i+j\leq 2}a_{ij}x^iy^j}{\sum_{0\leq i+j\leq 2}b_{ij}x^iy^j}.$$

Acta Math. Sinica **8** (1958), 23–35. (Chinese. German summary)

1430b:

Chin, Yuan-shun. On algebraic limit cycles of degree 2 of the differential equation

$$\frac{dy}{dx}=\frac{\sum_{0\leq i+j\leq 2}a_{ij}x^iy^j}{\sum_{0\leq i+j\leq 2}b_{ij}x^iy^j}.$$

Sci. Sinica **7** (1958), 934–945.

{The second article is a translation of the first.}

Consider the real differential system

$$(E_2)\quad \dot{x} = \sum_{0\le i+j\le 2} b_{ij}x^iy^j,\qquad \dot{y} = \sum_{0\le i+j\le 2} a_{ij}x^iy^j.$$

The most general such system, up to an affine change of variables

$$x_1 = \alpha x+\beta y+\varepsilon,\quad y_1 = \gamma x+\delta y+\eta \quad (\alpha\delta-\beta\gamma \ne 0),$$

which has an ellipse as a limit cycle, is

$$(E_2')\quad \dot{x} = y(ax+by+c)+x^2+y^2-1,\qquad \dot{y} = -x(ax+by+c),$$

where $c^2>a^2+b^2$ $(a\ne 0)$. Furthermore, the ellipse $(x^2+y^2=1)$ is the only periodic solution of (E_2'), and there are just three possible topological structures for (E_2') in the plane. These three possibilities are each structurally stable differential systems.

The author also investigates the qualitative form of the solutions of (E_2') in case $a=0$.

L. Markus (Minneapolis, Minn.)

1431:

Chin, Yuan-Shun. On algebraic limit cycles of degree 2 of the differential equation

$$\frac{dy}{dx} = \frac{\sum_{0\le i+j\le 2} a_{ij}x^iy^j}{\sum_{0\le i+j\le 2} b_{ij}x^iy^j}.$$

Sci. Record (N.S.) **1** (1957), no. 2, 1–3.

This is a summary of the paper with the same title reviewed above. *L. Markus* (Minneapolis, Minn.)

1432:

★Antosiewicz, H. A. A survey of Lyapunov's second method. Contributions to the theory of nonlinear oscillations, Vol. IV, pp. 141–166. Annals of Mathematics Studies, no. 41. Princeton University Press, Princeton, N.J., 1958. ix + 211 pp. $3.75.

The paper is an excellent and concise survey of the principal mathematical theorems underlying Lyapunov's second (direct) method which has been in recent years so highly developed. It is an important tool in the study of stability of systems and has found wide application in the USSR. The paper contains a good bibliography.

J. P. LaSalle (Baltimore, Md.)

1433:

Razumikhin, B. S. On the application of the Liapunov method to stability problems. J. Appl. Math. Mech. **22** (1958), 466–480 (338–349 Prikl. Mat. Meh.).

The system of the perturbed motion is the linear system $\dot{x}_i=\sum_{j=1}^n P_{ij}(t)x_j$ $(i=1,\cdots,n)$. The n^2 numbers P_{ij} are considered to be the coordinates of a point in n^2-dimensional space P. Let $V(x)$ denote a positive definite quadratic form with constant coefficients. $\dot{V}$ denotes its time derivative along solutions of the linear system. $L(V)$ denotes the region of points in P for which $\dot{V}$ is a non-positive quadratic form. Thus, by Liapunov's stability theorem, if the curve defined by $P_{ij}(t)$ lies in $L(V)$, the system is stable. The equation for the surface bounding $L(V)$ is $D_n=0$, where D_n is the discriminant of $\dot{V}$. This surface is an elliptic paraboloid with asymptotic directions given by the row vectors of the matrix defining $\dot{V}$. A single nth order equation is studied as a special case. Several applications of the results to the study of stability are given, and it is indicated how the results can be applied to determine stability of nonlinear systems.

J. P. LaSalle (Baltimore, Md.)

1434:

Sidériadès, Lefteri. Méthodes topologiques dans un espace à trois dimensions. C. R. Acad. Sci. Paris **247** (1958), 911–913.

The paper presents some general ideas on classification of critical points of a system $\dot{x}_i=x_i(x_1, x_2, x_3)$ $(i=1, 2, 3)$.

W. Kaplan (Ann Arbor, Mich.)

1435:

Matuda, Tizuko. On the behavior of the solutions of an ordinary differential equation of first order near a non-movable singularity. Sûgaku **8** (1956/57), 139–148. (Japanese)

A Japanese edition of the author's papers "Sur les points singuliers des équations differentielles ordinaires du premier ordre. I, II, III, IV, V" [Nat. Sci. Rep. Ochanomizu Univ. **2** (1951), 13–17; **4** (1953), 36–39; **5** (1954), 1–4, 175–177; **8** (1957), 1–6; MR **14**, 274; **15**, 126; **16**, 1023; **17**, 847; **20** #3319] together with some supplementary results.

H. Yamabe (Osaka)

1436:

Sobolevskiĭ, P. E. Generalized solutions of first order differential equations in Hilbert space. Dokl. Akad. Nauk SSSR **122** (1958), 994–996. (Russian)

This note states a sequence of results about the existence of generalized solutions of a first order, not necessarily linear, differential equation in Hilbert space.

M. M. Day (Urbana, Ill.)

1437:

Perov, A. I. On uniqueness theorems for ordinary differential equations. Dokl. Akad. Nauk SSSR **120** (1958), 704–707. (Russian)

The author announces several general results concerning the uniqueness of solutions to the initial value problem $x'=f(t, x)$, $x(0)=x_0$; where $x(t)$ and $f(t, x)$ are functions with values in a real Banach space. As this note contains no proofs, stating the results here would be equivalent to rewriting the note. So we will only remark that the uniqueness is split into two parts: (i) proving that for a certain $k\geqq 0$ the relation $x(t)\sim y(t)$ defined by

$$\lim_{t\to 0} \|x(t)-y(t)\|t^{-k} = 0$$

splits the solutions into equivalence classes consisting of one element only, and (ii) proving there is only one such equivalence class.

R. R. Kemp (Kingston, Ont.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 1624.

1438:

Levitan, B. M.; and Sargsyan, I. S. Asymptotic estimates of derivatives of eigenfunctions of the Schrödinger equation. Izv. Akad. Nauk Armyan. SSR. Ser.

Fiz.-Mat. Nauk **10** (1957), no. 5, 19–32. (Russian. Armenian summary)

If the function $q(x)$, defined on all of E_3, is bounded from below and approaches $+\infty$ as $|x|\to\infty$, it is well known that the operator $L=\Delta-q(x)$ on $L^2(E_3)$ admits only point spectrum. If the proper values are $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$ and the corresponding proper functions are $\psi_1(x), \psi_2(x), \cdots, \psi_n(x), \cdots$, the authors prove the following theorem: If at the point x, $q(x)$ admits derivatives of order $\alpha-1$, then as $\mu\to\infty$

$$\sum_{\mu<\mu_n\leq\mu+1}\{D_x{}^\alpha\psi_n(x)\}^2 = O(\mu^{2\alpha+2}),$$

where $\mu_n{}^2=\lambda_n$ and $D^\alpha=\partial^\alpha/\partial x_1{}^{\alpha_1}\partial x_2{}^{\alpha_2}\partial x_3{}^{\alpha_3}$ $(\alpha_1+\alpha_2+\alpha_3=\alpha)$.

R. R. Kemp (Kingston, Ont.)

1439:

Ždanovič, V. F. Solution by the Fourier method of non-selfadjoint mixed problems for hyperbolic systems on the plane. I. Mat. Sb. (N.S.) **47** (**89**) (1959), 307–354. (Russian)

This paper is the first part of a detailed study of the following problem: to solve the system

$$\partial u(x, t)/\partial t = A(x)[\partial u(x, t)/\partial x]+B(x)u(x, t)$$

$(0\leq x\leq l<\infty, 0\leq t\leq T<\infty)$ under the boundary condition

$$M[\partial u(0, t)/\partial t]+Nu(0, t)+P[\partial u(l, t)/\partial t]+Qu(l, t) = 0$$

and the initial condition $u(x, 0)=f(x)$, where $u(x, t)$ is an n-dimensional vector with complex coordinates, $A(x)$ and $B(x)$ are matrices satisfying certain regularity conditions, and M, N, P, Q are constant matrices. After carefully defining terms and developing a fundamental tool, the energy integral, the author proceeds to the formal treatment of the problem by the method of separation of variables in the form $y(x)e^{\lambda t}$. The eigenfunctions $y_s(x)$ correspond to the values λ_s of λ for which the system consisting of the differential equation and the boundary condition has non-zero solutions; the initial function $f(x)$ is expanded in terms of these eigenfunctions and the coefficients are determined by Fourier formulas. A major part of the paper is devoted to multiple eigenfunctions. All theorems are formulated only in the generality required by the above context and are established in the language of matrix theory. The second and third parts are to deal with regularity of the boundary conditions and with classical vs. generalized solutions. A previous note has briefly summarized some of the results [Dokl. Akad. Nauk SSSR **114** (1957), 934–937; MR **18**, 864].

R. N. Goss (San Diego, Calif.)

1440:

Pucci, Carlo. Discussione del problema di Cauchy per le equazioni di tipo ellittico. Ann. Mat. Pura Appl. (4) **46** (1958), 131–153. (English summary)

The first part is expository: Study of the Cauchy problem for the Laplace equation and its physical applications.

Main result of the second part: Positive solutions of Cauchy problems for the Laplace equation depend continuously on the data. The proof is based on a new estimation from above for the value $u(P)$ of a function harmonic on a sphere $\overline{P_0P}\leq r$ in terms of $u(P_0)$ and the normal derivative of u at a point on $\overline{P_0P}=r$, where u assumes its minimum.

A. Huber (Muenchenstein)

1441:

Mařík, Jan. Eine Bemerkung über elliptische Differentialgleichungen. Czechoslovak Math. J. **8** (**83**) (1958), 246–250. (Russian summary)

Let G be a bounded open region of E_m, H the boundary, $\mathscr{J}$ the system of elliptic operators J of the form

$$J = \sum_{i,j=1}^m a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{k=1}^m b_k\frac{\partial}{\partial x_k}$$

$(\sum_{i,j=1}^m a_{ij}(x)\tau_i\tau_j\geq 0$ for all $\tau_1, \cdots, \tau_m)$, and $\mathscr{L}$ the system of operators L of the form $L=c-J, J\in\mathscr{J}$, for which the following implication holds: $u\geq 0$ on H, $Lu\geq 0$ on $G\Rightarrow u\geq 0$ on G (every $L\in\mathscr{L}$ has the property that there exists at most one solution of the first boundary value problem for the equation $Lu=f$). The author proves the following. If $J\in\mathscr{J}$, $L=c-J$ and there exists a function v such that $v>0$ on $\bar{G}$ and $Lv>0$ on G, then $L\in\mathscr{L}$. Further he gives estimates for the solutions of $Lu=f$ where $L\in\mathscr{L}$ and generalizes a theorem of Morgenstern [J. Rational Mech. Anal. **5** (1956), 203–216; MR **17**, 1211; theorem 4].

M. Zlámal (Brno)

1442:

Oleĭnik, O. A. Solution of fundamental boundary value problems for second order equations with discontinuous coefficients. Dokl. Akad. Nauk SSSR **124** (1959), 1219–1222. (Russian)

Verfasser behandelt zunächst die elliptische Differentialgleichung

$$(*)\ \sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = 0;\quad a_{ij} = a_{ji}, x = (x_1, \cdots, x_n).$$

Die Koeffizienten $a_{ij}(x)$ sind hinreichend glatte Funktionen ihrer Argumente im Bereich Ω, eventuell mit Ausnahme von Punkten auf gewissen $(n-1)$-dimensionalen Mannigfaltigkeiten, in welchen diese Funktionen unstetig sein dürfen. Durch diese $(n-1)$-dimensionalen Mannigfaltigkeiten zerfällt Ω in eine endliche Anzahl von Teilgebieten Ω_j $(j=1, \cdots, m)$. Die Mannigfaltigkeit, welche die Grenze von Ω_k und Ω_l abgibt, wird mit S_{kl} bezeichnet. Von a_{ij} wird ferner die Existenz von Grenzwerten $a_{ij}{}^{(k)}$ und $a_{ij}{}^{(l)}$ (beiderseits von S_{kl}) vorausgesetzt. Zum Bereich Ω gehört die glatte Berandung S. Dann wird für (*) ein Dirichletsches Problem in folgendem Sinne gestellt: Gesucht wird eine in $\Omega+S$ stetige Funktion $u(x)$, welche der Gleichung (*) in allen Punkten der Bereiche Ω_j genügt sowie den Bedingungen

$$u|_S=f,\quad a_k du/dN_k=a_l du/dN_l$$

auf S_{kl}. Dabei ist f eine auf S vorgegebene Funktion, a_j $(j=1, \cdots, m)$ sind gewisse bestimmt vorgegebene Konstanten, d/dN bedeutet den Operator

$$\sum_{i,j=1}^n a_{ij}\cos(nx_i)\frac{\partial}{\partial x_j}$$

mit n als orientierter Normale zu S_{kl}. Eine solche Funktion $u(x)$ heisst eine Lösung des Dirichletschen Problems im klassischen Sinne. Daneben werden jetzt auch verallgemeinerte Lösungen $u(x)$ des Dirichletschen Problems betrachtet. Wieder muß eine solche Lösung der Bedingung $u|_S=f$ genügen. Dazu wird jetzt die Bedingung

$$(**)\quad \iint_\Omega\sum_{i,j=1}^n aa_{ij}\frac{\partial u}{\partial x_i}\frac{\partial F}{\partial x_j}\,d\Omega = 0$$

gestellt. Dabei bedeutet $F(x)$ eine beliebige Funktion der Klasse $W_2^1(\Omega)$, die auf S verschwindet und a eine Funktion über Ω, die in den Punkten von Ω_j die konstanten Werte a_j annimmt. Während nun die Eindeutigkeit der Lösung des klassischen Dirichlet-Problems aus dem Umstand hervorgeht, daß für diese Lösung das Maximumprinzip zu Recht besteht, wie Verfasser bei anderer Gelegenheit bewiesen hat, ergibt sich die Eindeutigkeit der Lösung des verallgemeinerten Dirichletschen Problems durch Differenzbildung aus (**) für zwei solche Lösungen u_1 und u_2 mit $F\equiv u_1-u_2$. Auf diesem Wege gelingt Verfasser die Konstruktion derartiger verallgemeinerter Lösungen. Weiterhin werden gewisse Eigenschaften dieser verallgemeinerten Lösungen und ihre Beziehungen zum klassischen Dirichletschen Problem studiert.

Ein weiterer Abschnitt der Arbeit ist Gleichungen von parabolischem Typus gewidmet. Insbesondere wird das erste Randwertproblem unter analogen Voraussetzungen behandelt. *M. Pinl* (Cologne)

1443:

Itô, Seizô. A remark on my paper "A boundary value problem of partial differential equations of parabolic type" in Duke Mathematical Journal. Proc. Japan Acad. **34** (1958), 463–465.

This note gives a modification of the argument used in the referenced paper [Duke Math. J. **24** (1957), 299–312; MR **19**, 864] to insure the continuity of the fundamental solution. *C. G. Maple* (Ames, Iowa)

1444:

Friedman, Avner. Remarks on the maximum principle for parabolic equations and its applications. Pacific J. Math. **8** (1958), 201–211.

Consider

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\partial^2 u/\partial x_i\partial x_j + \sum_{i=1}^{n} a_i(x,t)\partial u/\partial x_i + a(x,t)u - \partial u/\partial t$$

for $(x,t)=(x_1,\cdots,x_n,t)$ in the closure $\bar{D}$ of a given $(n+1)$-dimensional domain D, where $a(x,t)\leq 0$ and L is parabolic. The coefficients of L are assumed to be continuous in $\bar{D}$. The author proves the following generalization of Nirenberg's strong maximum principle [Comm. Pure Appl. Math. **6** (1953), 167–177; MR **14**, 1089]. Let u be continuous in $\bar{D}$ and let the derivatives of u which appear in Lu be continuous in D. Consider $P^0\in\partial D$ and let $t=\varphi(x)$ be the equation of ∂D near P^0, where $\varphi\in C''$, $\partial\varphi/\partial x_i|_{P^0}=0$, and D is on the side $t<\varphi(x)$. If $Lu\geq 0$ in D, u assumes its maximum M at P^0, $\lim_{P\to P^0}\partial u(P)/\partial x_i=0$, $\lim_{P\to P^0}\sum a_{ij}(P)\partial^2 u(P)/\partial x_i\partial x_j\leq 0$ $(P\in D)$, and

$$1+\sum a_{ij}\partial^2\varphi/\partial x_i\partial x_j|_{P^0}>0,$$

then $u\equiv M$ at all points $Q\in D$ which can be connected to P^0 by a simple continuous curve in D along which t is non-decreasing from Q to P^0. Examples of cases in which certain of the above conditions are not satisfied and in which the theorem is false are given. The author also extends the theorem of E. Hopf [Proc. Amer. Math. Soc. **3** (1952), 791–793; MR **14**, 280] and O. A. Oleĭnik [Mat. Sb. (N.S.) **30** (**72**) (1952), 695–702; MR **14**, 280] on the sign of the normal derivative at a positive maximum (necessarily on the boundary) of u such that $Lu\geq 0$, where L is a second order elliptic operator, to the operator

$$L'u \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\partial^2 u/\partial x_i\partial x_j + \sum_{i,j=1}^{m} b_{ij}(x,t)\partial^2 u/\partial t_i\partial t_j + \sum_{i=1}^{n} a_i(x,t)\partial u/\partial x_i + \sum_{i=1}^{m} b_i(x,t)\partial u/\partial t_i + a(x,t)u,$$

where L' is elliptic in x, parabolic in t, and $a(x,t)\leq 0$. The same result has also been proved by Vyborny [Dokl. Akad. Nauk SSSR **117** (1957), 563–565]. The author applies this result to obtain the uniqueness of the solution of the Neumann problem for $Lu=0$. The paper concludes with an application of Nirenberg's weak maximum principle to obtain a uniqueness theorem for certain nonlinear problems [a generalization of a result of Ficken, J. Rational Mech. Anal. **1** (1952), 573–578; MR **14**, 282].

D. G. Aronson (Minneapolis, Minn.)

1445:

Adler, György. Sulla caratterizzabilità dell'equazione del calore dal punto di vista del calcolo delle variazioni. Magyar Tud. Akad. Mat. Kutató Int. Közl. **2** (1957), 3/4, 153–157. (Hungarian and Russian summaries)

A proof that the heat equation is not equivalent to the Euler equation of a variational problem.

L. M. Graves (Chicago, Ill.)

1446:

Mlak, W. Limitation of solutions of parabolic equations. Ann. Polon. Math. **5** (1958/59), 237–245.

L'A. considera il sistema di m equazioni:

$$(*)\quad \frac{\partial z_s}{\partial t} = F_s\left(x,t,z_1,\cdots,z_m,\frac{\partial z_s}{\partial x_i}\frac{\partial^2 z_s}{\partial x_i\partial x_k}\right)\quad (s=1,2,\cdots,m).$$

(Nota: a secondo membro nell'equazione s-sima compaiono solo le derivate della funzione s-sima.) Si fa la seguente ipotesi: per tutti i numeri q_{ik}, tali che la forma

$$\sum_{i,k=1}^{n} q_{ik}\lambda_i\lambda_k$$

sia semidefinita negativa, si abbia

$$F_s(x,t;u_1,\cdots,u_m,0,q_{ik}) \leq \sigma_s\quad (t,u_1,\cdots,u_m)$$

dove le funzioni σ_s, continue, soddisfanno alla condizione:

$$u_i'\leq u_i''\ (i\neq 0),\ u_s'=u_s''\Rightarrow$$
$$\sigma_s(t,u_1',\cdots,u_m')\leq\sigma_s(t,u_1'',\cdots,u_m'').$$

In questa ipotesi le soluzioni del sistema (*) si possono limitare in relazione all'integrale massimo destro del sistema ordinario $y_s'=\sigma_s(t;y_1,\cdots,y_m)$. Questo teorema viene utilizzato dall'A. anche per maggiorare gli incrementi $u_s(x,t+h)-u_s(x,t)$ $(a\leq t\leq b,\ h>0)$ in base agli stessi incrementi calcolati per $t=a$. *G. Prodi* (Trieste)

1447:

Mlak, W. The first boundary value problem for a nonlinear parabolic equation. Ann. Polon. Math. **5** (1958/59), 257–262.

L'A. considera il primo problema al contorno per l'equazione

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}+f(x,t,u).$$

Dimostra un teorema di esistenza, utilizzando la tecnica di Leray-Schauder. Le maggiorazioni a priori vengono ottenute mediante un teorema di confronto con l'integrale massimo destro dell'equazione ordinaria $\partial u/dt=\sigma(t,\omega)$, dove la funzione continua $\sigma(t,\omega)$, è tale che $|f(x,t,u)|\leq \sigma(t,|u|)$.

Per risultati analoghi ved. G. Prodi, Rend. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17**, **86** (1953), 1–47 [MR **16**, 259]. *G. Prodi* (Trieste)

1448:

Durstine, R. M.; and Shaffer, D. H. Determination of upper and lower bounds for solutions to linear differential equations. Quart. Appl. Math. **16** (1958), 315–317.

The boundary value problem $L(u)=-\varphi$ for an elliptic operator L with the boundary conditions $M_j(u)=\lambda_j$ is considered. It is assumed that the Green's function for this differential system does not change sign.

It is shown that if w_1 and w_2 are functions satisfying the boundary conditions and having the properties that $L(w_1)$ does not change sign and that

$$1>M\geqq\{L(w_2)+\varphi\}/\{L(w_1)+\varphi\}\geqq m$$

then u lies between the two functions

$$u_1=w_1-(w_1-w_2)/(1-M),\quad u_2=w_1-(w_1-w_2)/(1-m).$$

H. F. Weinberger (College Park, Md.)

1449:

Danilyuk, I. I. On a problem involving a skew derivative for first order elliptic systems. Dokl. Akad. Nauk SSSR **122** (1958), 9–12. (Russian)

G sei ein Bereich in der z-Ebene, $\Gamma=\sum_{j=0}^{m}\Gamma_j$ eine geeignete Berandung von G. In G wird die Differentialgleichung

$$(1)\qquad \frac{\partial f}{\partial\bar z}=B(z)\overline{f(z)},\quad f=u+iv,\quad \frac{\partial}{\partial\bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

betrachtet, die einem System für die Funktionen u, v aequivalent ist. Auf $G+\Gamma$ wird eine komplexwertige stetige Funktion $f=u+iv$ gesucht, die in Γ eine allgemeine Lösung von Gleichung (1) darstellt. Sie soll sich nach Γ stetig fortsetzen lassen und zusammen mit der Ableitung $\dfrac{\partial f}{\partial z}$ auf Γ den Randbedingungen

$$(2)\qquad \operatorname{Re}\left[a\frac{\partial f}{\partial z}+bf\right]=\gamma,\quad \frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$$

genügen (Problem B). Dabei sind a, b und γ gegebene Funktionen auf Γ. Das Problem B ergibt sich aus dem Randwertproblem allgemein elliptischer Systeme erster Ordnung in der Ebene, wenn Randbedingungen der Gestalt

$$\alpha u_x+\beta u_y+\gamma v_x+\delta v_y+\nu u+\mu v=g$$

aufgestellt werden. Derlei Fragen wurden vor allem von I. N. Vekua untersucht [cf. I. N. Vekua, Mat. Sb. (N.S.) **31 (73)** (1952), 217–314; MR **15**, 230]. In der vorliegenden Arbeit wird ein anderer Zugang zur Behandlung dieser Fragen vorgeschlagen. Als besonderes Hilfsmittel zum Studium des Problems B erweisen sich dabei gewisse Hilfssätze, die zu den für elliptische Probleme vom Typus Riemann-Hilbert verwendeten im Falle von drei komplexen elliptischen Gleichungen erster Ordnung aequivalent sind, wenn in die Randbedingungen keine Ableitungen der unbekannten Funktionen eingehen. Verfasser gewinnt notwendige und hinreichende Bedingungen für die Lösbarkeit des Problems B. *M. Pinl* (Cologne)

1450:

Danilyuk, I. I. The use of Fredholm's system of equations in investigating a problem involving a skew derivative. Dokl. Akad. Nauk SSSR **122** (1958), 175–178. (Russian)

Im Gebiet D mit Rand Γ wird die Lösung der Differentialgleichung

$$(1)\qquad \frac{\partial f}{\partial\bar z}=\lambda B(z)\overline{f(z)},\quad \frac{\partial}{\partial\bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

gesucht, die nach Γ stetig fortgesetzt werden kann und zusammen mit ihrer Ableitung $\partial f/\partial z$ den Randbedingungen

$$(2)\qquad \operatorname{Re}\left[a\frac{\partial f}{\partial z}+\lambda bf\right]=\gamma,\quad \frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$$

genügt (Problem B_λ).—Problem B_λ erweist sich mit Problem C_λ aequivalent: man bestimme allgemeine Lösungen (stetige Lösungen) des Systems

$$(3)\qquad \frac{\partial F_1}{\partial\bar z}=\lambda B\bar F_1,\quad \frac{\partial F_2}{\partial\bar z}=\lambda B_z F_1+\lambda^2|B|^2F_1,\quad \frac{\partial F_3}{\partial\bar z}=\bar F_2,$$

die sich nach Γ stetig fortsetzen lassen und auf Γ den Randbedingungen

$$(4)\qquad \operatorname{Re}[g_1(t)F(t)]=h_1(t),$$

$$g_1=\begin{pmatrix}\lambda b & a & 0\\ 1 & 0 & -1\\ i & 0 & i\end{pmatrix},\quad F=\begin{pmatrix}F_1\\ F_2\\ F_3\end{pmatrix},\quad h_1=\begin{pmatrix}\gamma\\ 0\\ 0\end{pmatrix}$$

genügen. Sodann wird (4) durch

$$\operatorname{Re}[g(t)\varphi(t)]=h(t),$$

$$g(t)=\begin{pmatrix}0 & t^{-\kappa} & 0\\ 1 & 0 & -1\\ i & 0 & i\end{pmatrix},\quad h=\begin{pmatrix}\gamma e^{\omega_x}\\ 0\\ 0\end{pmatrix}$$

ersetzt und das System

$$\frac{\partial\varphi}{\partial\bar z}=A_\lambda\varphi+B_\lambda\bar\varphi,$$

$$A_\lambda=\begin{pmatrix}0 & 0 & 0\\ (\lambda^2|B|^2+\lambda\beta_z)e^{i\Omega} & 0 & 0\\ * & 0 & 0\end{pmatrix},$$

$$B_\lambda=\begin{pmatrix}\lambda B & 0 & 0\\ (\lambda^2\beta B+\lambda B_z)e^{i\Omega} & 0 & 0\\ -\lambda\bar\beta & e^{i\Omega} & 0\end{pmatrix}$$

behandelt.—Für die weitere Untersuchung sind die Fallunterscheidungen $\kappa\geqq 0$ und $\kappa_1=-\kappa<0$ für die Indexwerte κ der Probleme B_λ und C_λ wesentlich. In beiden Fällen gelangt der Verfasser zu einer Spektraltheorie der Probleme B_λ und C_λ. Dabei werden die Integraloperatoren

$$T_\lambda\varphi\equiv-\frac{1}{\pi}\iint_D\left[\frac{\omega(t)}{t-z}+G(z)\frac{\overline{z\omega(t)}}{1-tz}\right]d\sigma_t,$$

$$\omega(t)\equiv A_\lambda\varphi+B_\lambda\bar\varphi$$

und

$$K_\lambda\varphi \equiv -\frac{1}{\pi}\iint_D\left[\frac{\omega(t)}{t-z}+\overline{G(t)}\,\frac{\overline{\omega(t)}}{\bar{t}(1-\bar{t}z)}\right]d\sigma_t$$

und die Integralgleichungen

$$\varphi(z) = T_\lambda\varphi(z)+\Phi(z) \text{ bzw. } \varphi(z) = K_\lambda\varphi(z)+\Phi(z)$$

von Bedeutung. *M. Pinl* (Cologne)

1451:

Vekua, I. N. Über die korrekte Stellung der Riemann-Hilbertschen Aufgabe. Ann. Acad. Sci. Fenn. Ser. A. I. no. 251/10 (1958), 14 pp.

Die Riemann-Hilbertsche Aufgabe besteht im Aufsuchen der stetigen Lösungen $w(z)$ des folgenden Randwertproblems:

$$\partial_{\bar{z}}w+Aw+B\overline{w} = F \quad \text{in } G,$$

$$\mathscr{R}[\overline{\lambda(z)}\cdot w(z)] = \gamma(z) \quad \text{auf } \Gamma.$$

Dabei sind A, B, F gegebene Funktionen im $m+1$-fach zusammenhängenden Gebiete G, λ und γ gegebene Funktionen auf dem Rand $\Gamma=\Gamma_0+\Gamma_1+\cdots+\Gamma_m$ von G; $\partial_{\bar{z}}=\frac{1}{2}(\partial/\partial x+i\partial/\partial y)$. Der Zuwachs von $(1/2\pi)$ arg $\lambda(z)$ bei positivem Umlauf des Gebietsrandes heisst der Index n der Aufgabe.

Diese Aufgabe A ist auch bei geeigneten Regularitätsvoraussetzungen über die Bestimmungselemente nicht für alle n lösbar bzw. die Lösung ist nicht eindeutig bestimmt (d.h. die Aufgabe ist nicht korrekt gestellt). Für $n\geqq m$ werden nun zusätzlich die Werte der Lösung $w(z)$ für eine gewisse Anzahl von Punkten in G und auf dessen Rand beliebig vorgeschrieben und für die neue Aufgabe B Existenz, Eindeutigkeit und stetige Abhängigkeit der Lösung von den Bestimmungselementen bewiesen. Für $n<m$ wird ein eindeutig lösbares Problem angegeben, dessen Lösung eine gewisse Anzahl von Polen besitzt, und es werden notwendige und hinreichende Bedingungen für die stetige Lösbarkeit der Aufgabe A aufgestellt.

K. Strebel (Fribourg)

1452:

Roždestvenskiĭ, B. L. Discontinuities of solutions of quasi-linear equations. Mat. Sb. (N.S.) **47** (89) (1959), 485–494. (Russian)

A peculiarity which distinguishes some, but not all, systems of quasilinear equations from linear is that solutions of the former may exhibit discontinuities even in the presence of continuous initial data. This paper undertakes to determine a class of quasilinear systems which have this property.

The system is

$$(*) \qquad \partial u_i/\partial t+\sum_{j=1}^{2} a_{ij}(\partial u_j/\partial x) = b_i \quad (i = 1, 2),$$

restricted to be hyperbolic, in which the coefficients a_{ij}, b_i are continuous and twice continuously differentiable functions of t, x, u_1, u_2. Characteristics $dx/dt=\xi_1(r, s, t, x)$, where $r(u_1, u_2, t, x)$ and $s(u_1, u_2, t, x)$ are Riemann invariants of the system, are called contact characteristics if they can be lines of strong discontinuity of the solution. The property of having a contact characteristic is an intrinsic invariant of the system. The system (*) is called weakly nonlinear if each of the characteristics $dx/dt=\xi_1$ and $dx/dt=\xi_2$ is a contact characteristic. The following theorem is proved: in order that the system (*) not lead to the formation of a discontinuity in the solution under arbitrary continuous initial data, it is necessary and sufficient that this system be weakly nonlinear.

R. N. Goss (San Diego, Calif.)

1453:

Èĭdel'man, S. D. A solution of some boundary problems for parabolic systems. Černivec. Derž. Univ. Nauk. Zap. Ser. Fiz.-Mat. Nauk **4** (1952), no. 2, 123–137. (Russian)

Consider the system of differential equations

$$(1) \qquad \frac{\partial u_k}{\partial t} = \sum_{j=1}^{n} a_{kj}\frac{\partial^2 u_j}{\partial x^2} \quad (k = 1, \cdots, n),$$

where (a_{kj}) is a real $n\times n$ non-singular constant matrix. The author studies the following two boundary problems. Problem A: Solve (1) in $\{x>x_0, -\infty<t<\infty\}$ subject to $u_k(x_0, t)=f_k(t)$ $(k=1, \cdots, n)$; and Problem B: Solve (1) in $\{0<x<x_0, -\infty<t<\infty\}$ subject to $u_k(0, t)=f_{1,k}(t)$ and $u_k(x_0, t)=f_{2,k}(t)$. It is assumed that there exist constants b_k such that $\sum_{k,j=1}^{n} a_{kj}b_k\alpha_k\alpha_j$ is positive definite (α_j real) and that the characteristic roots of (a_{kj}) are not pure imaginary. Let M denote the maximum of the orders of the elementary divisors of $(a_{kj})-\lambda I$. If the $f_k(t)$ have $[(M+1)/2]+3$ continuous derivatives such that

$$\int|tf_k^{(j)}(t)|dt<\infty,$$

then problem A, subject to the additional conditions $\lim_{t\to\pm\infty} u_k(x, t)=\varphi(x)$ uniformly in x, $|u_k|<D(1+x)^{-(1+\epsilon)/2}$, and $\partial u_k/\partial x$ bounded for $x\geqq x_0$ is uniformly correctly set. A similar result holds for problem B. The proof of these results is by Fourier transform methods and makes use of the results of N. I. Simonov [Dokl. Akad. Nauk SSSR **44** (1944), 259–261; Moskov. Gos. Univ. Uč. Zap. **100**, Mat. **1** (1946), 53–84; MR **6**, 228; **14**, 652] for elliptic systems.

D. G. Aronson (Minneapolis, Minn.)

1454:

Èĭdel'man, S. D. Liouville theorems and theorems on stability for solutions of parabolic systems. Mat. Sb. N.S. **44** (86) (1958), 481–508. (Russian)

This paper is a continuation of previous work by the author on parabolic equations, and depends both for methods and notation on his earlier papers. In the present work theorems of Liouville type and stability theorems are proved for systems of the form

$$(1) \quad \frac{\partial^{n_i}\mu_i}{\partial t^{n_i}} = \sum_{j=1}^{N}\sum_{k_0 2b+\sum k_i\leq n_j\cdot 2b} A_{ij}^{(k_0 k_1\cdots+k_n)}(t)\,\frac{\partial^{k_0+k_1+\cdots k_n}\mu_j}{\partial t^{k_0}\partial x_1^{k_1}\cdots\partial x_n^{k_n}} \qquad (i = 1, 2, \cdots, N),$$

which are parabolic in the sense of Petrowski. The results are obtained from the Green's representation of the solution and derive accordingly from estimates on the Green's matrix $G(t, \tau, x)$ associated with (1). This matrix is said to satisfy a condition Λ_1^+ $[\Lambda_2^+]$ if, for all t, τ, $t>\tau$, $x_1, \cdots, x_n$ in the half space $\tau\geqq T$, the inequalities

$$\Lambda_1 \qquad |D^m G(t, \tau, x)| \leqq C_m[a(t, \tau)]^{K_m}e^{-c\left|\frac{x}{a}\right|^q}$$

or

$$\Lambda_2 \qquad |D^m G(t, \tau, x)| \leqq C_m[a(t, \tau)]^{K_m}e^{-b(t,\tau)-c\left|\frac{x}{a}\right|^q}$$

are satisfied for suitable continuous $a(t, \tau)$, $b(t, \tau)$ which are monotone increasing in t, $a(\tau, \tau)=b(\tau, \tau)=0$, and positive constants C_m, c. If the inequality $\tau\geqq T$ is replaced

by the inequality $t \leq T$, then $G(t, \tau, x)$ is said to satisfy a condition Λ_1 $[\Lambda_2^-]$. Such conditions are shown to be satisfied in cases of interest, and form the basis for general theorems on behavior of solutions.

Examples: 1. Suppose Λ_1^- satisfied with $\lim_{\tau\to-\infty} a(t, \tau) = \infty$, $\lim_{m\to\infty} K_m = -\infty$. Then any solution of (1) in the half space $t \leq T$ which satisfies

$$\left|\frac{\partial^k \mu_i}{\partial t^k}\right| \leq c[1+|x|]^\beta \quad (k = 0, 1, \cdots, n_i - 1)$$

is necessarily a system of polynomials of degree not exceeding $[\beta]$.

2. The trivial solution of (1) is called E_K-stable if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any solution $\mu(x, t)$ of a Cauchy problem in $t \geq 0$ satisfying $|\mu(x, 0)| \leq \delta e^{K|x|}$ one has always $|\mu(x, t)| \leq \varepsilon e^{K|x|}$. The author shows that if Λ_1^+ is satisfied for (1) with $m = 0$ and $a(t, \tau)$ bounded, and if $K_0 + n > 0$, then the trivial solution of (1) is E_0-stable.

Various related theorems are also proved and many particular cases, including certain non-linear equations, are discussed in detail. *R. Finn* (Stanford, Calif.)

1455:

Lions, J. L. Sur les problèmes mixtes pour certains systèmes paraboliques dans des ouverts non cylindriques. Ann. Inst. Fourier, Grenoble **7** (1957), 143–182.

Let Ω be an open set in $R_x^n \times R_t^1$, $t > 0$, and let $A = A_x = \sum (-1)^{|p|} D_x^p (a_{pq}(x, t) D_x^q)$ be an elliptic differential operator in $x = (x_1, \cdots, x_n)$ of $2m$ order. The author treats the equation $Au + \partial u/\partial t = f$ by preassigning the value $u(x, 0)$ as well as the values of u and its derivatives up to $(m-1)$ order along the boundary of Ω not situated in $t = 0$. He begins with the preparation of necessary function spaces, and proves a general existence theorem which reads as follows.

Let F be a Hilbert space with $\mathscr{D}(\Omega) \subset F \subset \mathscr{D}'(\Omega)$ such that the injections ($\mathscr{D}(\Omega) \to F$ and $F \to \mathscr{D}'(\Omega)$) are continuous and $\mathscr{D}(\Omega)$ is dense in F by the norm $\|f\|$ of F. Let $E(f, h)$ be linear in $f \in F$ and anti-linear in $h \in H = \mathscr{D}(\Omega)$. For fixed $h \in H$, let $E(f, h)$ be continuous in $f \in F$, and real part of $E(h, h) > \alpha\|h\|^2$ for $h \in H$. Let, further, $h \to E(f, h)$ be continuous in $\mathscr{D}(\Omega)$ by Schwartz's topology of $\mathscr{D}(\Omega)$. Thus $E(f, h) = \langle Df, h\rangle$ with $Df \in \mathscr{D}'(\Omega)$, the parenthesis denoting the duality between $\mathscr{D}(\Omega)$ and $\mathscr{D}'(\Omega)$. Then, for any distribution $T \in F'$, there exists $u \in F$ such that $Du = T$.

Detailed discussions are given how to apply this existence theorem to the mixed problem of the parabolic equation $Au + \partial u/\partial t = f$; the uniqueness and the stability of the solutions are also discussed. The results are extended to the case of system of equations.

K. Yosida (Tokyo)

1456:

Fage, M. K. The Cauchy problem for Bianchi's equation. Mat. Sb. N.S. **45** (87) (1958), 281–322. (Russian)

The results of this paper were in part described in the author's abstract [Dokl. Akad. Nauk SSSR **108** (1956), 780–783; MR **18**, 214]. Let F be a closed domain in euclidean n-space E_n; for each subset K of the set N of all positive integers not greater than n let p_K be a continuous function defined on F and let $D_K = \prod_{i\in K} \partial/\partial t_i$, where t_i is the ith coordinate variable in E_n; the linear differential operator $B = \sum_K p_K D_K$ and the equation $Bu = h$, h a given function defined in F, are the subjects of this paper; the author proves that certain differentiability properties on the coefficients p_K and on the initial values suffice that the Cauchy problem have a unique solution and that stronger conditions suffice that it be given by an integral formula here called Riemann's formula. *M. M. Day* (Urbana, Ill.)

1457:

Smirnov, M. M. Dirichlet's problem for an equation of mixed type. Vestnik Leningrad. Univ. **14** (1959), no. 1, 130–133. (Russian. English summary.)

"This paper treats the problem of finding a solution $u = u(x, y)$ of the equation

$$\frac{\partial^4 u}{\partial x^4} + 2 \operatorname{sgn} y \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0$$

in the plane domain $y > -h$ ($h > 0$) and satisfying the boundary conditions

$$u(x, -h) = \varphi_0(x), \quad u_y(x, -h) = \varphi_1(x) \quad (-\infty < x < +\infty).$$

It is proved that there exists one and only one solution of this problem." (Author's summary)

A. Huber (Muenchenstein)

1458:

Pucci, Carlo. Studio di un sistema di equazioni differenziali della dinamica dei gas. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 653–657.

A Dirichlet problem is considered for the pair of equations,

$$T(U_{xx} + U_{yy}) - U_x T_x - U_y T_y = 0,$$
$$T_{xx} + T_{yy} - \alpha U_{xx} - \alpha U_{yy} = 0$$

with $\alpha \neq 0$ a constant, in the domain B with the boundary conditions $U = \psi$, $T = \phi$ on the boundary of B. It is shown first that the solution satisfies a certain maximum principle but is not necessarily unique. If ϕ vanishes on the boundary a solution need not exist. Finally it is proved that if: (1) ϕ is purely positive or negative, (2) the boundary is of class $C^{(3,\lambda)}$ in the sense of Miranda, (3) ϕ and ψ are of class $C^{(3)}$; then there exists a solution U, T in the class $C^{(2,\lambda)}$ in the closure of B. The proof is based on a theorem of Bernstein-Schauder-Cacciopoli and uses the results of Bers and Nirenberg to find the necessary a priori estimates.

The functions U and T are the rates of diffusion and temperature of a gas in a diffusion chamber.

C. S. Morawetz (New York, N.Y.)

1459:

Bakel'man, I. Ya. The first boundary value problem for some non-linear elliptic equations. Dokl. Akad. Nauk SSSR **124** (1959), 249–252. (Russian)

Existence and uniqueness of the solution of the first boundary value problem is proved for a class of equations of the form $F(r, s, t, x, y) = g(x, y)$ for the domain $x^2 + y^2 \leq R^2$. The considered class includes, e.g., the equations

$$(r+t)^3 - 3(r+t)(rt - s^2) + (r+t) = g(x, y),$$
$$(r+t)^5 - 5(r+t)^3(rt - s^2) + 5(r+t)(rt - s^2)^2 + (r+t) = g(x, y)$$

and the linear elliptic equation

$$a(x, y)r + b(x, y)s + c(x, y)t = d(x, y).$$

A. Huber (Muenchenstein)

POTENTIAL THEORY

See also 1440, 1832.

1460:

Huber, Alfred. Zum Randverhalten subharmonischer Funktionen. Compositio Math. **13** (1958), 257–262.

Littlewood proved: Let $v(z)$ be subharmonic in $|z|<1$ and $\int_{-\pi}^{\pi} |v(re^{i\theta})|d\theta = O(1)$, then for almost all θ, $\lim_{r\to1} v(re^{i\theta}) = v(e^{i\theta})$ exists. The author proves the theorem: For almost all θ,

$$\lim_{\tau\to0} v(e^{i\theta}-\tau e^{i(\theta+\chi)}) = v(e^{i\theta})$$

for almost all χ ($|\chi|<\pi/2$). The theorem was proved by Tsuji [Comment. Math. Univ. St. Paul. **5** (1956), 3–16; MR **18**, 122] under a certain condition.

M. Tsuji (Tokyo)

1461:

Delsarte, J.; et Lions, J. L. Moyennes généralisées. Comment. Math. Helv. **33** (1959), 59–69.

The authors consider the following problem: Given a singular hyperbolic operator

$$D = -\sum \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)+\frac{\partial^2}{\partial t^2}+\frac{1}{t}\,q(t)\,\frac{\partial}{\partial t}+r(t)$$

and some positive number a, find the functions f defined in R^n such that, if $u(x, t)$ is a solution of

$$Du = 0,\quad u(x, 0) = f(x),\quad \frac{\partial}{\partial t}u(x, 0) = 0,$$

then $u(x, a)=f(x)$. The authors show that this problem is equivalent to "un problème de moyenne périodicité à une variable" and use this result to obtain a proof for the following theorem, first stated by Delsarte [C. R. Acad. Sci. Paris **246** (1958), 1358–1360; MR **20** #2548]: Let f be an infinitely differentiable function defined in R^n, let $u(x, r)$ denote the mean value of f taken over the sphere with center at x and radius r, and let a and b denote two fixed positive numbers. If $u(x, a)=u(x, b)=f(x)$ in R^n, then $f(x)$ is harmonic. The argument requires that, when $n>3$, exception be made for a finite number of values of a and b which are independent of the function f.

F. W. Gehring (Helsinki)

1462:

Nečas, Jindřich. Solution du problème biharmonique pour le coin infini. I, II. Časopis Pěst. Mat. **83** (1958), 257–286, 399–424. (Czech. Russian and French summaries)

The author employs the theory of the Mellin transform to prove the existence and uniqueness of a function biharmonic in the interior of an infinite convex wedge, the function and its normal derivative assuming, in a specially prescribed sense, specific boundary values. The function is then expressed explicitly in terms of these boundary values with the aid of Green's functions, the latter being given both tabularly and graphically to facilitate numerical work. Properties of the solution in the neighborhood of the wedge boundary are studied in detail, a principal aid in this study being a detailed development of the solution of the biharmonic problem for a half-plane.

J. F. Heyda (Cincinnati, Ohio)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See 1463, 1622.

SEQUENCES, SERIES, SUMMABILITY

1463:

Bajraktarević, M. Sur une généralisation de certaines suites itérées. Acad. Serbe Sci. Publ. Inst. Math. **12** (1958), 27–38.

The iterated sequences considered here are analogous to those treated earlier by the author [see, e.g., Srpska Akad. Nauka. Zbornik Radova **35**. Mat. Inst. **3** (1953), 61–74; Naučno Društvo NR Bosne i Hercegovine. Djela **4**, Odjeljenje Privredno-Tehničkih Nauka **1** (1954); Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 13–22; MR **15**, 784; **17**, 949]. It is shown that their limits in certain cases satisfy functional equations reducible to Abel or Schröder form. The precise definitions and statements of theorems are too involved for resumé here.

I. M. Sheffer (University Park, Pa.)

1464:

Lee, K. P. The transformed form of Cauchy's inequality. Acta Math. Sinica **7** (1957), 340–345. (Chinese)

1465:

Mikolás, Miklós. Sur un problème d'extrémum et une extension de l'inégalité de Minkowski. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **1** (1958), 101–106.

The extremal values of

$$\sum_{\mu=1}^{m}\left(\sum_{\nu=1}^{n} x_{\mu\nu}{}^{r}\right)^{p}$$

for positive $x_{\mu\nu}$, p and r subject to $\sum_{\mu=1}^{m} x_{\mu\nu}=c_\nu$ are considered by using Lagrange's multipliers. The value found, $n^{1-pr}(\sum_{\nu=1}^{n} c_\nu{}^{r})$, is a maximum if $r\leqq1$ and $pr\leqq1$ and a minimum if $r\geqq1$, $pr\geqq1$. Minkowski's inequality results when $p=1/r$.

Similar results for double integrals are obtained.

A. J. Macintyre (Cincinnati, Ohio)

1466:

Kuttner, B. The problem of "translativity" for Hausdorff summability (addendum). Proc. London Math. Soc. (3) **9** (1959), 318–320.

By means of an easy computation, the author obtains as an addendum to his previous paper [same Proc. **6** (1956), 117–138; MR **17**, 359]: each multiplicative Hausdorff transformation is translative for bounded sequences.

G. G. Lorentz (Syracuse, N.Y.)

1467:

Ramanujan, M. S. On products of summability methods. Math. Z. **69** (1958), 423–428.

Ein früheres Ergebnis des Verf. [Math. Z. **65** (1956), 442–447; MR **18**, 573], nach dem $AB\supset A$ gilt für A=Abelverfahren, B=Quasi-Hausdorff Verfahren und beschränkte Folgen (für diesen Satz wird ein neuer Beweis gegeben), wird auf eine grössere Klasse von Folgen

ausgedehnt. Ferner dürfen für B auch reguläre Verfahren

$$t_n = \sum_{k=0}^{\infty} \binom{n+k}{k} \Delta^k \mu_{n+1} s_k$$

eingesetzt werden, falls die Klasse der s_n passend eingeschränkt wird. *A. Peyerimhoff* (Giessen)

1468:

Obrechkoff, N. Sur la sommabilité absolue par les moyennes typiques des séries lacunaires. C. R. Acad. Bulgare Sci. **11** (1958), 1–4. (Russian summary)

Ist $\sum u_n$ eine $|R, \lambda, 1|$-summierbare Lückenreihe mit $u_n=0$ für $n_\mu<n<m_\mu$, $\mu=1, 2, \cdots$, $\lambda_{m_\mu}/\lambda_{n_\mu}\geqq q>1$, so ist

$$\sum_{\mu=1}^{\infty} \left| \sum_{n=m_\mu}^{n_{\mu+1}} u_n \right| < \infty.$$

A. Peyerimhoff (Giessen)

1469:

★Orlicz, H. W. Funktionalanalysis und allgemeine Theorie der linearen Transformationen. Colloque sur la théorie des suites, tenu à Bruxelles du 18 au 20 Décembre 1957, pp. 131–147. Centre Belge de Recherches Mathématiques. Librairie Gauthier-Villars, Paris; Établissements Ceuterick, Louvain; 1958. 167 pp. 220 fr. belges.

Seit Mazur, Banach und Orlicz haben funktionalanalytische Methoden immer breitere Verwendung in der Limitierungstheorie gefunden. Umgekehrt hat auch die Limitierung Anregungen zur Entwicklung funktionalanalytischer Begriffe wie "B_0-Raum" gegeben, weil viele Wirkfelder sich zwar nicht als B-Raum, aber doch als B_0-Raum auffassen lassen. Die Theorie der B_0-Räume liefert unter anderem Verträglichkeitsaussagen und Inäquivalenzsätze für Limitierungsverfahren. Bei der Untersuchung der Limitierbarkeit beschränkter Folgen ist es jedoch vorteilhafter, Saks-Räume zu benützen: Wird in der Einheitskugel $X_s=\{\|x\|\leqq 1\}$ eines B-Raumes eine zweite Norm $\|x\|^*$ betrachtet, bezüglich derer sich X_s als vollständiger metrischer Raum erweist, so nennt man X_s einen Saks-Raum. Beispiel: Menge T_b der beschränkten Folgen $x=\{t_n\}$, wobei $\|x\|=\sup_n |t_n|$ und $\|x\|^*=\sum 2^{-n}|t_n|$. In diesem Raum ist auch die (von jetzt an vorausgesetzte) Bedingung (Σ) erfüllt: Zu x_0 und $\rho>0$ gibt es ein $\delta>0$, so daß jedes x mit $\|x\|^*<\rho$ in der Form $x=x_1-x_2$ darstellbar ist, wo $\|x_1-x_0\|^*<\delta$ und $\|x_2-x_0\|^*<\delta$. Gewisse Sätze aus der Theorie der B-Räume lassen sich sinngemäß auf Saks-Räume übertragen. Bei der Behandlung von Folgen linearer Funktionale spielt dabei die Konstante

$$\mu=\lim_{\rho\to 0} \sup_{n=0,1,\cdots} \sup_{\|x\|^*\leqq\rho} |\xi_n(x)|$$

eine Rolle. Für die Konvergenzmenge C der $\xi_n(x)$ gilt (Satz A bis D): C ist entweder von erster Kategorie oder gleich X_s. Im Falle $\mu=0$ und $C\neq X_s$ ist C nirgends dicht und abgeschlossen. Ist $C=X_s$, so gilt $\mu=0$.

Mit diesen Hilfsmitteln beweist Verfasseren Schurschen Satz über konvergenzerzeugende Matrizen, einen Verträglichkeitssatz, einen Inäquivalenzsatz von Brudno über die Vereinigung von Matrixverfahren, und eine Nichtseparabilitätsaussage. Schließlich weist er auf eine von Mazur vorgeschlagene Klassifikation der Limitierungsverfahren hin. K_0 bedeutet die Menge der Verfahren A der Gestalt $A(x)=A\text{-}\lim x=a_1t_1+\cdots+a_mt_m$. Sind die Klassen K_β mit $\beta<\alpha<\Omega$ (Ordinalzahlen) erklärt, so besteht K_α aus den Verfahren der Form $A(x)=\lim_n A_n(x)$ mit $A_n\in K_{\alpha_n}$, wo $\alpha_n<\alpha$. Es enthält dann K_1 genau die zeilenfiniten Verfahren, während zu K_2 auch schon gewisse Nichtmatrixverfahren gehören.

K. Zeller (Tübingen)

1470:

Orlicz, W. On the summability of bounded sequences by continuous methods. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 549–556.

Eine Folge $x=\{t_n\}$ heißt Φ-limitierbar, wenn (a) $\phi(\tau; x)=\sum_{\nu=1}^{\infty} \varphi_\nu(\tau)t_\nu$ für $\tau\in\langle 0, T)$ existiert und (b) $\lim_{\tau\to T-}\phi(\tau; x)$ vorhanden ist. Die φ_ν seien dabei stetig in $\langle 0, T)$. Mit ϕ^* beziehungsweise $\phi_0{}^*$ bezeichnet Verfasser das Wirkfeld und das Nullwirkfeld des Verfahrens, mit ϕ^0 die Menge der (a) erfüllenden x, für die $\phi(\tau; x)$ in $\langle 0, T)$ beschränkt ist; T_b, T_c, T_0 bedeuten die Menge der beschränkten, konvergenten, gegen Null konvergenten Folgen. Das Verfahren ϕ heißt stetig, wenn aus (a) die gleichmäßige Konvergenz der betrachteten Reihe in jedem Intervall $\langle 0, t\rangle$ (wo $0<t<T$) folgt. Man faßt ϕ^0 und ϕ^* in üblicher Weise als B_0-Räume auf. Satz: Für ein stetiges konvergenztreues Verfahren ϕ sind folgende Eigenschaften gleichbedeutend: (a) Die Menge T_c ist abgeschlossen in ϕ^*; (b) die Menge T_b ist abgeschlossen in ϕ^0; (c) es gibt ein r und ein $c>0$, so daß $\sup_{\langle 0,T)}|\sum_{\nu=1}^{p} \varphi_{\nu+r}(\tau)\lambda_\nu|\geqq c \sup|\lambda_n|$ gilt für beliebige $\lambda_1, \lambda_2, \cdots, \lambda_p$; (d) es gibt ein r' und ein $c'>0$, so daß $\sup_{\langle 0,T)}|\sum_{\nu=1}^{\infty} \varphi_{\nu+r'}(\tau)\lambda_\nu|\geqq c' \sup|\lambda_n|$ gilt für jede beschränkte Folge λ_n; (e) ϕ^* enthält keine beschränkten divergenten Folgen. (In dem Satz sind noch zwei weitere äquivalente Bedingungen genannt.) Frühere Untersuchungen über Verfahren, die keine beschränkten divergenten Folgen limitieren, benützten beim Beweis auch nicht-funktionalanalytische Hilfsmittel [siehe Mazur und Orlicz, Studia Math. **14** (1954), 129–160; MR **16**, 814; Wilansky und Zeller, Trans. Amer. Math. Soc. **78** (1955), 501–509; MR **16**, 690]. Verfasser gelingt es, diesen Schönheitsfehler zu beseitigen. *K. Zeller* (Tübingen)

APPROXIMATIONS AND EXPANSIONS

1471:

Natanson, G. I. An interpolation process. Leningrad. Gos. Ped. Inst. Uč. Zap. **166** (1958), 213–219. (Russian)

The author studies Lagrange interpolation on the zeros $0<x_1<\cdots<x_n<\pi$ of the eigenfunctions $U_n(x)$ of the Sturm-Liouville problem

$$U''(x)+[\lambda^2-B(x)]U(x) = 0,$$

$$U'(0)-hU(0) = 0, \quad U'(\pi)+HU(\pi) = 0.$$

Let $B(x)$ be continuous and of bounded variation on $[0, \pi]$. Set

$$L_n{}^{SL}[f; x] = \sum_{k=1}^{n} \frac{U_n(x)}{(x-x_k)U_n'(x_k)} f(x_k).$$

Then for any $a\in(0, \pi/2)$ we have uniformly on $[a, \pi-a]$,

$$f(x)-L_n{}^{SL}[f; x] = O(\ln n)[\omega(f; n^{-1})+n^{-1}\|f\|],$$

where $\omega(f, \delta)$ is the modulus of continuity of f and $\|f\|=\max_{0\leqq x\leqq\pi}|f(x)|$. The term $O(\ln n)$ depends only on $B(x)$, L, H, and a. *P. J. Davis* (Washington, D.C.)

1472:

Stojaković, Mirko. Sur une formule d'interpolation par polynomes. Univ. Beogradu. Godišnjak Filozof. Fak. Novom Sadu **3** (1958), 241–252. (Serbo-Croatian summary)

This is a derivation of the Lagrange interpolation polynomial by means of a matrix inversion, developed in terms of symmetric functions involving either n or all of the $n+1$ abscissae concerned, namely, x_i, $i=o(1)n$, with or without x_j omitted.

An algorithm for building the inverse of the matrix

$$V_n = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & x_i & x_i^2 & \cdots & x_i^n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}$$

is given, and numerical values are tabulated for the elements of V_n^* in the case $x_i=i+1$, where V_n^* is V_n^{-1} with signs omitted, and the jth column multiplied by $(n-j+1)!(j-1)!$ to give integer values.

J. C. P. Miller (Cambridge, England)

1473:

★**Schmetterer, L. Sur les sommes riemanniennes.** Colloque sur la théorie des suites, tenu à Bruxelles du 18 au 20 Décembre 1957, pp. 109–118. Centre Belge de Recherches Mathématiques. Librairie Gauthier-Villars, Paris; Établissements Ceuterick, Louvain; 1958. 167 pp. 220 fr. belges.

This is a comprehensive report on the present status of the problem of the convergence as $h\to 0+$ of approximating sums $h\sum f(\lambda_i h)d_i$ to the integral

$$\int_0^\infty f(x)dx = \lim \int_\varepsilon^N f(x)dx.$$

Here $0<\lambda_1\leqq l_1\leqq \lambda_2\leqq \cdots$, $l_i\nearrow\infty$, $d_i=l_i-l_{i-1}$, $l_0=0$, $f(x)\in L(\varepsilon, N)$ for all ε, N, $0<\varepsilon\leqq N<\infty$, and the limit is taken as $\varepsilon\to 0$, $N\to\infty$. This problem was first studied by T. J. I'a. Bromwich and G. H. Hardy [Quart. J. Math. **39** (1908), 222–240] and has various developments. There are sketches of simple proofs of known results and of solutions to problems already stated, and indications of new results and new problems. {The author has pointed out several errata, e.g. factors 2π omitted on p. 116–117.}

John Todd (Pasadena, Calif.)

1474:

Stancu, D. D. Sur l'intégration numérique des fonctions de deux variables. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Ştí. Mat. **9** (1958), no. 1, 5–21. (Romanian. Russian and French summaries)

Let x_i, y_k ($i=1, 2, \cdots, m$; $k=1, 2, \cdots, n$) be the coordinates of a set of distinct points inside the rectangle $D(a\leqq x\leqq b,\ c\leqq y\leqq d)$.

Using the values of the function $f(x, y)$ and of some of its partial derivatives at these points, one may establish formulae of mechanical quadratures. More precisely, let $f(x, y)$ be defined and sufficiently often continuously differentiable in D and, for integral r_i, s_k, assume the values

$$\left(\frac{\partial^{\nu+\mu} f(x, y)}{\partial x^\nu \partial y^\mu}\right)_{\substack{x=x_i\\ y=y_k}} = f_{\nu\mu}(x_i, y_k)$$

to be known for $0\leqq\nu\leqq r_i-1$, $0\leqq\mu\leqq s_k-1$. Then

$$(*)\quad \iint_D f(x, y)dxdy = \sum_{i=1}^m \sum_{\nu=0}^{r_i-1} \sum_{k=1}^n \sum_{\mu=0}^{s_k-1} C_{i\nu k\mu} f_{\nu\mu}(x_i, y_k) + R[f].$$

The author sets himself and solves the following two problems. (a) The integers r_i, s_k and the values of $f_{\nu\mu}(x_i, y_k)$ being given at an arbitrary set (x_i, y_k), to determine the coefficients $C_{i\nu k\mu}$ and the remainder $R[f]$, so that (*) should have the (in general best possible) "degree of exactness" $(M-1, N-1)$, where $M=\sum_{i=1}^m r_i$, $N=\sum_{k=1}^n s_k$. (b) Selecting the set of points (x_i, y_k) in special ways, one may increase the degree of exactness of (*) and obtain quadrature formulae of Gauss' type

$$(**)\quad \iint_D f dx dx = \sum_{i,\nu,k,\mu} A_{i\nu} B_{k\mu} f_{\nu\mu}(x_i, y_k) + R_1[f].$$

Here x_i, y_k have to be selected so that $\int_a^b g(x)x^i dx = \int_c^d h(y)y^k dx = 0$ for $1\leqq i\leqq m-1$, $1\leqq k\leqq n-1$, with

$$g(x) = \prod_{i=1}^m (x-x_i)^{r_i},\quad h(y) = \prod_{k=1}^n (y-y_k)^{s_k}.$$

The corresponding degree of exactness is $(M+m-1, N+n-1)$. The proofs of both formulae use approximations by interpolation polynomials, some previous work of the author [Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 209–216; MR **20** #4917], that of (**) also some results of L. Tchakaloff (Bŭlgar. Akad. Nauk. Izv. Mat. Inst. **1** (1954), no. 2, 67–82; MR **16**, 1005] and T. Popoviciu [Acad. R. P. Romîne. Fil. Iaşi Stud. Cerc. Şti. **6** (1955), 29–57; MR **19**, 64]. The expressions obtained for the coefficients and the remainder terms are too complicated to be quoted here. {There are a few non-trivial misprints.}

E. Grosswald (Princeton, N.J.)

1475:

Stancu, D. D. Une méthode de construction des formules de quadrature d'un degré élevé d'exactitude. Com. Acad. R. P. Romîne **8** (1958), 349–358. (Romanian. Russian and French summaries)

Soient (a, b) un intervalle compact ou non, $f(x)$ et $p(x)$ intégrables sur (a, b), $p(x)\geqq 0$. Admettons qu'il existe pour $n=0, 1, 2, \cdots$ l'intégrale $\int_a^b p(x)x^n dx = c_n$ $(c_0>0)$.

A l'aide de la formule d'interpolation de Lagrange-Hermite on obtient une formule de quadrature pour $\int_a^b p(x)f(x)dx$. En utilisant des conditions supplémentaires, on obtient des formules de quadrature de degré d'exactitude plus élevé et on construit effectivement plusieurs formules de quadrature, mais elles sont trop compliquées pour être données ici. On retrouve, entre autres, certaines formules de Gauss, E. B. Christoffel, G. Szegö, Cavalieri-Simpson et T. Popoviciu.

S. Marcus (Bucarest)

1476a:

Hsu, L. C. A new type of polynomials approximating a continuous or integrable function. Studia Math. **18** (1959), 43–48.

1476b:

Hsu, L. C.; and Hsu, L. P. On a new class of approximating polynomials for real functions. Sci. Record (N.S.) **3** (1959), 65–70.

The authors consider the polynomials

$$(n\pi)^{-1/2}\sum_{k=0}^{n} f(k/n)[1-(k/n-x)^2]^n$$

and their various generalizations and prove theorems corresponding to known ones for Bernstein polynomials. The reviewer remarks that the main theorem of the first paper appears in Landau, *Einführung in die Differenzial- und Integralrechnung* [Noordhoff, Groningen, 1934; p. 106].
G. G. Lorentz (Syracuse, N.Y.)

1477:

Matsuyama, Noboru; and Takahashi, Shigeru. On the gap sequence having random signs. Sci. Rep. Kanazawa Univ. **6** (1958), 9–14.

Let $f(x)$ denote a Borel-measurable function with $\int_0^1 f(x)dx=0$; $\int_0^1 f^2(x)dx=1$, which is periodic with period 1. Let $\{n_k\}$ denote a lacunary sequence of increasing positive integers. Let $\{\varphi_k(t)\}$ denote the Rademacher system of orthogonal functions. Then the authors prove: If for a constant $\alpha>1$ and for $h\to 0$

$$\int_0^1 |f(x)-f(x+h)|^4 dx = O(|\log_2|h||^{-4\alpha}),$$

then for almost all t:

$$\limsup_{N\to\infty}\sum_{k=1}^{N}\frac{f(n_kx)\varphi_k(t)}{(2N\log\log N)^{1/2}} = 1 \quad \text{a.e. in } x.$$

And: If $|f(x)|\leq M$ and for $\alpha>0$, $h\to 0$

$$\int_0^1 |f(x)-f(x+h)|dx = O(|\log|h||^{-\alpha}),$$

then for almost all t

$$\lim_{N\to\infty}\left|\left(x;\, N^{-1/2}\sum_{k=1}^{N} f(n_kx)\varphi_k(t)\leq y\right)\right| = (2\pi)^{-1/2}\int_{-\infty}^{y} e^{-u^2/2}du.$$

The proofs are similar to that of Salem and Zygmund [Acta Math. **91** (1954), 241–301; MR **16**, 467].
J. F. Koksma (Amsterdam)

1478:

Watari, Chinami. On generalized Walsh Fourier series. Tôhoku Math. J. (2) **10** (1958), 211–241.

Let α be a fixed positive integer greater than 1 and let $\alpha(n)$ be for each $n=0, 1, 2, \cdots$ an integer satisfying $2\leq\alpha(n)\leq\alpha$. Let H_n be the cyclic group of order $\alpha(n)$ and let $H=H_0\times H_1\times H_2\times\cdots$ (infinite direct product). H is then a compact Abelian group. The present paper is essentially a very complete study of the expansions of functions on H in terms of the characters of H. There is a natural measure preserving map of H onto the unit interval [0, 1], and using this the theory may be rephrased as a study of the expansion of functions on [0, 1] by generalized Walsh-Rademacher series. This identification is due in the case $\alpha=2$ to N. J. Fine [Trans. Amer. Math. Soc. **65** (1949), 372–414; MR **13**, 126]. The larger part of the paper is taken up with a study of the L^p properties of this expansion, generalizing the basic results of R. E. A. C. Paley [Proc. London Math. Soc. **34** (1932), 241–279] as well as their extension by the reviewer [Mem. Amer. Math. Soc. no. 15 (1955); MR **17**, 257]. These both refer to the case $\alpha=2$. Earlier work on the case $\alpha>2$ is due to H. E. Chrestenson [Pacific J. Math. **5** (1955), 17–31; MR **16**, 920].
I. I. Hirschman, Jr. (St. Louis, Mo.)

1479:

van der Corput, J. G. Asymptotic developments. II. Generalization of the fundamental theorem on asymptotic series. J. Analyse Math. **5** (1956/57), 315–320.

[For part I, see same J. **4** (1955/56), 341–418; MR **18**, 890.] In this paper three formulations of the fundamental theorem in asymptotic series are given. The last formulation reads as follows. Theorem 3: Assume that h_0 is a fixed integer ≥ 0, that q_h $(h=h_0, h_0+1, \cdots)$ denote fixed real numbers with $q_h\leq q_{h+1}$ such that $q_h\to\infty$ as $h\to\infty$, and that for each fixed integer $h\geq h_0$ the order relation $a_h(\omega)=O(|\omega|^{-q_h})$ holds for the elements ω of an unbounded set Ω lying in the complex plane or in a Riemann surface. Then there exists a function $f(\omega)$ such that for the elements ω of Ω the order relation

$$f(\omega) = a_0(\omega)+a_1(\omega)+\cdots+a_{h-1}(\omega)+O(|\omega|^{-q_h})$$

holds for each fixed integer $h\geq h_0$.

In the applications of the methods in asymptotics sometimes more general results than those formulated above are needed. For this reason the author deduces in this paper a similar result where the function $|\omega|^{-q_h}$ is replaced by a much more general function, viz.: Theorem 4: Let h_0 denote a fixed integer ≥ 0. Let $\eta_h(\omega)$ $(h\geq h_0)$ denote positive functions of ω, defined for all the elements ω of Ω with sufficiently large absolute value. Assume that for each fixed integer $h\geq h_0$, $\eta_{h+1}(\omega)=O(\eta_h(\omega))$, and that for each fixed integer $h\geq h_0$ it is possible to find at least one fixed integer $r>h$ such that $\eta_r(\omega)=o(\eta_h(\omega))$. If, for each fixed integer $h\geq h_0$, $a_h(\omega)=O(\eta_h(\omega))$, then there exists a function $f(\omega)$ such that for the elements ω of Ω and for each fixed integer $h\geq h_0$

$$f(\omega) = a_0(\omega)+a_1(\omega)+\cdots+a_{h-1}(\omega)+O(\eta_h(\omega)).$$

For $f(\omega)$ can be chosen the sum $a_0(\omega)+a_1(\omega)+\cdots+a_{H-1}(\omega)$, where H denotes an integer ≥ 0 which depends on ω and which tends sufficiently slowly to infinity as $|\omega|\to\infty$ (ω in Ω). The particular case $\eta_h(\omega)=|\omega|^{-q_h}$ gives theorem 3.
S. C. van Veen (Delft)

FOURIER ANALYSIS

See also 1478, 1637, 1644, 1645, 1646, 1648.

1480:

Matsuyama, Noboru. On the convergence of some gap series. Sci. Rep. Kanazawa Univ. **6** (1958), 1–7.

Let $f(x)$ be L^2-integrable and periodic with period 1, such that

$$\int_0^1 f(x)dx = 0, \quad \int_0^1 f^2(x)dx = 1,$$

with Fourier series $\sum_{k=-\infty}^{+\infty} c_k e^{2\pi ikx}$. Put $R^2(n)=2\sum_{k>n}|c_k|^2$. The author considers the gap series

$$(1) \qquad \sum_{k=1}^{\infty} (L_p(k))^{-1} f(n_kx)$$

for a given sequence of positive integers $n_1<n_2<\cdots$, where

$$L_0(x) = x^{1-\alpha}(\log x)^\beta, \quad L_1(x) = x(\log x)^{1-\alpha}(\log_2 x)^\beta,$$
$$L_2(x) = x(\log x)(\log_2 x)^{1-\alpha}(\log_3 x)^\beta, \cdots,$$

etc., $\alpha>0$ and $\beta>1$ being constants. Then o.a. it is proved: If for any $\alpha>0$, $p\geqq 1$, $f(x)$ satisfies $R(n)=O((\log_p n)^{-\alpha})$, then (1) converges for almost all x. Similar results are proved in case that the sequence $\{n_k\}$ is lacunary; these results are related to those of Kac, Salem and Zygmund [Trans. Amer. Math. Soc. **63** (1948), 235–243; MR **9**, 426], Trumi [Tôhoku Math. J. **3** (1951), 89–103; MR **14**, 868], Kac [Bull. Amer. Math. Soc. **55** (1949), 641–665; MR **11**, 161], whereas the method used is that of Koksma [e.g. Indag. Math. **12** (1950), 354–367; J. Math. Pures Appl. **35** (1956), 289–296; MR **12**, 86; **18**, 380].

J. F. Koksma (Amsterdam)

1481:

Flett, T. M. Some more theorems concerning the absolute summability of Fourier series and power series. Proc. London Math. Soc. (3) **8** (1958), 357–387.

Im Anschluss an frühere Arbeiten [T. M. Flett, dieselber Proc. **7** (1957), 113–141, 211–218; MR **19**, 266] führt der Verf. hier durch die Forderung

$$\sum n^{k\gamma-1}|\tau_n{}^\alpha|^k<\infty,\quad \tau_n{}^\alpha=\binom{n+\alpha}{n}^{-1}\sum_{\nu\leqq n}\binom{n-\nu+\alpha-1}{n-\nu}\nu a_\nu,$$

eine Summierbarkeit $|C,\alpha,\gamma|_k$ und durch

$$\int_0^1(1-x)^{k-k\gamma-1}|\phi'(x)|^k dx<\infty$$

($\phi(x)=\sum a_nx^n$ konvergent für $|x|<1$) eine Summierbarkeit $|A,\gamma|_k$ für $\sum a_n$ ein. Es gelten die folgenden Vergleichssätze: Für $k\geqq 1$, $\alpha>\gamma-1\geqq -1$, $\beta>-1$, ist $|C,\alpha,\gamma|_k\subseteqq|C,\beta,\gamma|_r$, $r\geqq k$, $\beta\geqq\alpha+k^{-1}-r^{-1}$ $(k>1)$, $\beta>\alpha+1-r^{-1}$ $(k=1)$; $|C,\alpha,\gamma|_k\subseteqq|C,\beta,\mu|_k$, $\gamma>\mu$, $\beta\geqq\alpha-\gamma+\mu$; $|C,\alpha,\gamma|_k\subseteqq|C,\beta,\mu|_r$, $k>r\geqq 1$, $\gamma>\mu$, $\beta>\alpha-\gamma+\mu$; $|C,\alpha,\gamma|_k\subseteqq|A,\gamma|_k$. Weitere Sätze beziehen sich auf $|C,\alpha,\gamma|_k\subseteqq(C,\delta)$. Für Fourierreihen von Funktionen aus L^p ist die $|C,\alpha,0|_k$-Summierbarkeit genau dann eine Eigenschaft die lokal von der Funktion abhängt, wenn gilt: (i) für $k=1:1\leqq p<2$, $\alpha>1/p$ oder $p\geqq 2$, $\alpha>\frac{1}{2}$; (ii) $1<k<2:p=1$, $\alpha>1$ oder $1<p\leqq k$, $\alpha\geqq 1/p$ oder $k<p<2$, $\alpha>1/p$ oder $p\geqq 2$, $\alpha>\frac{1}{2}$; (iii) $k\geqq 2:p=1$, $\alpha>1$ oder $1<p<k'$ $(k^{-1}+k'^{-1}=1)$, $\alpha\geqq 1/p$, oder $p\geqq k'$, $\alpha\geqq 1/k'$. Ist $\phi(\rho e^{i\theta})=\sum c_n(\rho e^{i\theta})^n$ und wird $\tau_n{}^\alpha(\theta)$ von $\sum c_ne^{in\theta}$ gebildet, so gilt

$$(*)\quad \int_{-\pi}^{+\pi}\{\sum n^{k\gamma-1}|\tau_n{}^\alpha(\theta)|^k\}^{\lambda/k}d\theta\leqq B\int_{-\pi}^{+\pi}\left\{\int_0^1(1-\rho)^{k-k\gamma-1}|\phi'(\rho e^{i\theta})|^kd\rho\right\}^{\lambda/k}d\theta$$

für $\lambda\geqq k\geqq 1$, $0\leqq\gamma\leqq 1$, $\alpha>\operatorname{Max}(1/k,1/k')$. Aus (*) ergeben sich Aussagen über $|C,\alpha,\gamma|_k$-Summierbarkeit p.p. als Folge aus Voraussetzungen über $\phi(z)$ (als Anwendung werden Lipschitzfunktionen betrachtet). Es werden verschiedene zu (*) ähnliche Ungleichungen behandelt. Schliesslich werden Integrale wie sie in (*) auf der rechten Seite auftreten (auch mit α-fachem Integral ϕ_α von ϕ statt ϕ) durch Integrale über die Randwerte von ϕ und ϕ_α abgeschätzt.

A. Peyerimhoff (Giessen)

1482a:

Tomić, M. Sur la sommation des séries de Fourier de fonctions continues. II. Bull. Acad. Serbe Sci. (N.S.) **20** Cl. Sci. Math.-Nat. Sci. Math. **3** (1957), 33–40.

1482b:

Tomić, M. Sur la sommation des séries de Fourier de fonctions continues. II. Glas Srpske Akad. Nauka **228** Od. Prirod.-Mat. Nauka (N.S.) **13** (1957), 61–74. (Serbo-Croatian. French summary)

[Part I appeared in Publ. Inst. Math. Acad. Serbe Sci. **10** (1956), 19–36; MR **18**, 574. A note in the French paper points out that, although these papers have identical titles, their contents are different.] The author is concerned with conditions on a triangular matrix which will make it sum (as a matrix of convergence factors) the Fourier series of every function with a prescribed modulus of continuity. The following theorem appears in both papers: Let f have modulus of continuity ω and let

$$(*)\quad \int_\delta^{2\pi}t^{-2}\omega(t)dt=O(\delta^{-1}\omega(\delta))$$

(French paper) or (*) $\omega(\delta)>O(1/|\log(1/\delta)|)$ (Serbian paper). If

$$(\mathrm{A})\quad \omega(n^{-1})\int_0^{2\pi}\left|d\left\{\sum_{k=1}^n k^{-1}\lambda_{kn}\sin kt-\tfrac{1}{2}(\pi-t)\right\}\right|\to 0$$

and

$$(\mathrm{B})\quad n\omega(n^{-1})\int_0^{2\pi}\left|\sum_{k=1}^n k^{-1}\lambda_{kn}\sin kt-\tfrac{1}{2}(\pi-t)\right|dt\to 0,$$

then if f has Fourier coefficients (a_k,b_k) we have

$$\frac{1}{2}a_0+\sum_{k=1}^n\lambda_{kn}(a_k\cos kx+b_k\sin kx)\to f(x).$$

The author then seeks to replace condition (B) by one bearing more explicitly on the λ_{kn}. In the French paper he gives the alternative condition: for some positive ε, $|\lambda_{kn}-1|=O(n^{-\theta})$ for $k=1,2,\cdots,[n^\varepsilon]$, where $\theta=\theta(\varepsilon)$, and $\lambda_{kn}\downarrow 0$. In the Serbian paper he gives the alternative condition

$$\lambda_{k,n}=1-P_n{}^{-1}\sum_{\nu=1}^n p_{n-k+\nu},\quad p_\nu\geqq 0,\quad P_n=\sum_{\nu=0}^n p_\nu\to\infty,$$

and

$$n^{-1}(p_0+p_1+\cdots+p_n)\geqq\cdots\geqq\tfrac{1}{2}(p_{n-1}+p_n)\geqq p_n.$$

The French paper also contains a general theorem which implies (after a change of variable) the first theorem quoted: Let $\{g_n(t)\}$ be a sequence of continuous functions of bounded variation and of period 1, with $g_n(0)=0$, $g_n(1)=1$. Then the linear functionals $U_n(\lambda,x)=\int_0^1 x(t+\lambda)dg_n(t)$ converge weakly to $x(\lambda)$, whose modulus of continuity is ω, if $\omega(n^{-1})\int_0^1|dg_n(t)|\to 0$ and $n\omega(n^{-1})\int_0^1|g_n(t)-1|dt\to 0$. The following lemma is of independent interest: If ω satisfies (*) and x_n is a trigonometric polynomial of order n such that $|f(t)-x_n(t)|\leqq C\omega(n^{-1})$, then $|x_n'(t)|\leqq C_1n\omega(n^{-1})$; this generalizes a theorem of Zamansky's [C. R. Acad. Sci. Paris **224** (1948), 704–706; MR **8**, 457] which in turn generalizes a classical theorem of S. Bernstein's.

R. P. Boas, Jr. (Evanston, Ill.)

1483:

Noble, M. E. On a convergence criterion of Hardy and Littlewood. Quart. J. Math. Oxford Ser. (2) **9** (1958), 28–39.

Soit $f\in L$, sur $[-\pi,\pi]$, et soit s_m la m-ième somme partielle de Fourier de f. Soit $\varphi(t)$ une fonction continue croissante $(t\geqq 0)$ avec $\varphi(0)=0$, $\varphi(t)=O\{(\log 1/|t|)^{-1}\}$

($t \to 0+$). Supposons que: (1) $\int_0^t |f(x+u)-f(x)|du = o\{t\varphi(t)\}$ ($t \to 0$), (2) il existe une suite d'entiers $n_k \to \infty$ et une suite $\lambda_{n_k} = o(n_k)$ telles que

$$\int_{n_k^{-1}}^{\lambda_{n_k}^{-1}} u^{-1}\varphi(u)du = O(1),$$

$\liminf_k [\min \{s_m(x) - s_n(x)\}] \geqq 0$ ($|m, n-n_k| \leqq \lambda_{n_k}, m > n$). Dans ces conditions $s_{n_k}(x) \to f(x)$ ($k \to \infty$).

S. Mandelbrojt (Paris)

1484:

Tureckiĭ, A. H. On the saturation class in Hölder's method of summing Fourier series. Dokl. Akad. Nauk SSSR **121** (1958), 980–983. (Russian)

Let C denote the continuous functions of period 2π. Let $\gamma = \{\gamma_k^n\}$ ($k = 1, \cdots, n$; $n = 1, 2, \cdots$) be a method of summing Fourier series:

$$P_n^\gamma(x) = a_0/2 + \Sigma_1^n \gamma_k^n (a_k \cos kx + b_k \sin kx).$$

Let $\phi(n)$ decrease to zero. Definition: The process γ is saturated with respect to ϕ if the following two conditions are satisfied:

(i) $\max |P_n^\gamma(x) - f(x)| > b\phi(n) \quad (n = 1, 2, \cdots)$

for all $f \in C$, where b is a number depending on f but not on n;

(ii) $\max |P_n^\gamma(x) - f(x)| = O(\phi(n))$

for at least one f in C. The saturation class of γ is the set of all those f in C for which (ii) holds.

Theorem: The method H_r ($r > 0$) of Hölder is saturated with respect to the function $\phi(n) = (\log n)^{r+1}/n$. The author determines the saturation classes in case $r = 1$ or 2. This paper is based on earlier work of Zamansky [Ann. Sci. École Norm. Sup. (3) **66** (1949), 19–94; **67** (1950), 161–198; MR **11**, 27; MR **12**, 328], and of Favard [Bull. Sci. Math. (2) **61** (1937), 209–224, 243–256].

A. Shields (Ann Arbor, Mich.)

1485:

Takeda, Kazuaki. On the Abel summability of multiple Fourier series by spherical means. Tôhoku Math. J. (2) **10** (1958), 253–261.

A comparison is made between the behaviour of

$$P(\varepsilon, m) = \int_0^{1/\varepsilon} u^{k-1}[u^{2m+2}+1]^{-(k+1)/2} f_x(\varepsilon u)du \qquad (m > -(k+1)^{-1}),$$

$$E(\varepsilon, n, \alpha) = \int_0^{1/\varepsilon} u^{k-\alpha-3} e^{-u^{-n-1}} f_x(\varepsilon u)du \qquad (n > -1, \alpha \geqq k-1)$$

as $\varepsilon \to 0$, when $f_x(u)$ is the mean of $f(x) = f(x_1, x_2, \cdots, x_k)$, periodic 2π in each variable, over the sphere with centre x and radius u. The conclusion is that $E \to 0$ implies $P \to 0$, or conversely, according as $n > m$ or $m > n$. Further, if $\alpha \geqq k-1$, the bounded variation of E implies that of P, or conversely, according as $n > m$ or $m > n$.

These conclusions extend results of Levinson [Duke Math. J. **2** (1936), 138–146] and Sunouchi [Tôhoku Math. J. (2) **1** (1950), 167–185; MR **12**, 174].

H. R. Pitt (Nottingham)

1486:

Konyuškov, A. A. Convergence of certain series of Fourier coefficients. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 189–196. (Russian)

Let $f \in L^p(0, 2\pi)$, $1 \leqq p < \infty$, have the Fourier series $a_0/2 + \sum_{n=1}^\infty a_n \cos nx + b_n \sin nx$. Let $E_n(f)$, $n = 0, 1, \cdots$, denote the best approximation of f in the L^p norm by trigonometric polynomials of degree $\leqq n$. The author seeks conditions for the convergence of (*) $\sum_{n=1}^\infty n^\gamma(|a_n|^\beta + |b_n|^\beta)$, $\beta > 0$, $-1 \leqq \gamma < \infty$, in terms of $E_n(f)$. Theorem: A condition on the sequence of positive numbers φ_n necessary and sufficient that (*) converge for all f satisfying $E_n(f) = O(\varphi_n)$ is (**) $\sum_{n=1}^\infty n^{\gamma-\beta/p'}(\varphi_n^*)^\beta < \infty$ where $\varphi_n^* = \min_{m \leqq n} \varphi_m$. Here $1 \leqq p \leqq 2$, $0 < \beta \leqq 1$, $\gamma - \beta/p' > -1$. If $p = \infty$, $0 < \beta \leqq 1$, $\gamma - \beta/2 > -1$ the condition (**), with p' replaced by 2, is also necessary and sufficient. Very special cases of the theorem have been established previously by S. Bernstein and S. Stečkin. A similar theorem is stated involving, instead of $E_n(f)$, the L^p modulus of continuity of f of the kth order. *R. R. Goldberg* (Evanston, Ill.)

1487:

Timan, A. F. On Jackson's theorems. Ukrain. Mat. Ž. **10** (1958), 334–336. (Russian.)

The author proves the following theorem. Let

$$F(x) = (\pi^{-1}) \int_{-\pi}^{\pi} f(t) D_{r,s}(x-t)dt,$$

where $r > 0$ and $D_{r,s}(t) = \sum_1^\infty \cos(kt - \frac{1}{2}\pi)/k^r$, and $\|f\|_p < \infty$. Then as $n \to \infty$ we have $E_n^*(F)_p = O[(1/n^r)\omega_k(f, 1/n)_p]$. Here $E_n^*(f)_p = \inf \|f - T_n\|_p$, taken over all trigonometric polynomials of degree at most n, and

$$\omega_k(f, t)_p = \sup \left\| \sum_0^k (-1)^\nu \binom{k}{\nu} f(x+\nu h) \right\|_p$$

taken over all $|h| \leqq t$. This generalizes a result of S. M. Nikol'skiĭ [Trudy Mat. Inst. Steklov **15** (1945); MR **7**, 435].

A. Shields (Ann Arbor, Mich.)

1488:

Kahane, Jean-Pierre; et Rudin, Walter. Caractérisation des fonctions qui opèrent sur les coefficients de Fourier-Stieltjes. C. R. Acad. Sci. Paris **247** (1958), 773–775.

Soit F une fonction à valeurs complexes définie sur la droite réelle. On sait que si $F(f(x)) \in A$ (classe des fonctions sommes de séries de Fourier absolument convergentes) pour toute $f \in A$ à valeurs réelles, F est analytique sur R [Katznelson, mêmes C. R. **247** (1958), 404–406; MR **20** #4152]. De même si $\{F(a_n)\}$ est une suite de coefficients de Fourier pour toute suite $\{a_n\}$ de coefficients de Fourier réels, F est analytique à l'origine [Helson et Kahane, ibid. **247** (1958), 626–628; MR **20** #4737]. Les auteurs démontrent maintenant le théorème suivant: Si $\{F(c_n)\}$ est une suite de coefficients de Fourier-Stieltjes pour toute suite $\{c_n\}$ de coefficients de Fourier-Stieltjes, F est une fonction entière.

S. Mandelbrojt (Paris)

1489:

Kahane, Jean-Pierre. Sur les fonctions moyenne-périodiques bornées. Ann. Inst. Fourier, Grenoble **7** (1957), 293–314.

C désignant l'espace des fonctions continues sur la droite (avec la topologie de la convergence uniforme), $f \in C$

est dite moyenne-périodique (m.p.) si $\tau(f)$ (le sous-espace vectoriel fermé engendré par les translatées de f) ne coïncide pas avec C. L'auteur donne quelques propriétés de ces fonctions, ainsi que celles des distributions m.p. (qu'on obtient en remplaçant dans la définition précédente C par $\mathscr{D}'$—espace des distributions sur la droite). Ainsi, en appelant spectre de f la suite de λ, chacun compté autant de fois qu'il y a d'exponentielles-monômes $x^p e^{i\lambda x} \in \tau(f)$ (ce qui permet d'introduire la série de Fourier, $f \sim \sum a(\lambda, p) x^p e^{i\lambda x}$), on démontre qu'une condition nécessaire et suffisante pour qu'une distribution m.p. soit bornée est que le spectre soit réel et simple et que $a(\lambda) = O(|\lambda|^N)$ pour N assez grand $(f \sim \sum a(\lambda) e^{i\lambda x})$; ou encore que cette distribution soit p.p. Schwartz. Si $f \in C$, et si $\tau(f)$ ne contient que des fonctions bornées, f est dite C-pseudo-périodique (C-ps.p.). On désigne par $Q(\Lambda)$ une condition nécessaire et suffisante que doit satisfaire une suite réelle Λ pour que toute fonction m.p. de spectre contenu dans Λ soit C-ps.p. $Q(\Lambda)$ équivaut alors à l'existence d'un intervalle I et d'une constante K tels que, pour toute somme finie $s = \sum_{\lambda \in \Lambda} a(\lambda) e^{i\lambda x}$, on ait $|s| < K|s|_I$ $(|f|_I = \sup_{x \in I} |f(x)|)$; ou encore, à la condition suivante: toute fonction uniformément approchable sur I par des polynômes $\sum_{\lambda \in \Lambda} a(\lambda) e^{i\lambda x}$ est prolongeable de façon unique sur la droite en une fonction, m.p. bornée de spectre contenu dans Λ. Dans le cas où C est remplacé par E^2 (espace des fonctions localement $\in L^2$, avec convergence L^2 sur tout compact) et lorsque le spectre est contenu dans une suite régulière donnée $(\inf_{\lambda, \lambda' \in \Lambda} |\lambda - \lambda'| > 0)$, on a, pour I assez grand, l'équivalence des normes $\|f\|$, $\|f\|_I$ et $(\sum |a(\lambda)|^2)^{1/2}$; et, en appelant pseudo-période la quantité l_2 qui est la borne inférieure des longueurs des I pour lesquels cette proprieté a lieu, on a $l_2 = 2\pi\Delta$, où Δ est la densité supérieure de répartition de Λ. On étudie du même point de vue le cas $\mathscr{D}'$. On étudie, enfin, les fonctions moyenne-périodiques bornées, sommes de séries de Fourier absolument convergentes.

S. Mandelbrojt (Paris)

1490:

Kunze, R. A. An extension of a theorem of Mandelbrojt. Proc. Amer. Math. Soc. **9** (1958), 553–557.

L'auteur généralise à n dimensions un théorème de Mandelbrojt concernant le "prolongement" dans un demi-plan d'une fonction $f \in L_\infty$, satisfaisant aux conditions $F * K = 0$ $(K \in L_1)$, la transformée de K ne s'annulant pas sur une demi-droite [J. Math. Pures Appl. (9) **35** (1956), 211–222; MR **18**, 305]. Désignons par $\mathfrak{G}$ l'espace euclidien à n dimensions considérée comme un groupe vectoriel. Le dual d'un cône C dans $\mathfrak{G}$ est le cône $\hat{C}$ dans le groupe dual $\hat{\mathfrak{G}}$ composé de tous les x tels que $(x \cdot t) \geqq 0$ pour tout t dans C. Désignons par $\Gamma(\hat{C})$ l'ensemble de tous les vecteurs (complexes) $x + iy$ avec $x \in \hat{C}$, $y \in \hat{\mathfrak{G}}$, et par $S(C)$ l'ensemble de tous les $v + iy$, où v est le sommet de C et $y \in \hat{\mathfrak{G}}$. La généralisation en question s'énonce alors de la manière suivante: Soit $F \in L_\infty(\hat{\mathfrak{G}})$, et soit C un cône fermé, convexe dans $\mathfrak{G}$ de sommet à l'origine, et dont l'intérieur n'est pas vide. Soit a un élément dans C. Supposons qu'à chaque $t \notin C - a$ corresponde une fonction $K \in L_1(\hat{\mathfrak{G}})$ telle que $K * F \equiv 0$, avec $k(t) \neq 0$, où k est la transformée (inverse) de Fourier de K. Il existe alors une function F_0 analytique sur l'intérieur de $\Gamma(C)$, "prolongeant" F (c'est-à-dire telle que, quelle que soit la suite $x_n \to v = 0$, $x_n \in \hat{C}^0$, C^0 intérieur de C, il existe une suite extraite, x_{n_i}, telle que $F_0(x_{n_i} + iy) \to F(v + iy)$ p.p.) et possédant les propriétés suivantes: $|F_0(x + iy)| \leq \|F\|_\infty e^{a \cdot x}$ pour tout $x \in \hat{C}^0$, et $y \in \hat{\mathfrak{G}}$;

$$\lim_{x \to 0} \int_D |F_0(x + iy) - F(y)|^2 dy = 0$$

lorsque $x \to 0$ dans $\hat{C}^0$, D étant un compact quelconque contenu dans $S(C)$.

S. Mandelbrojt (Paris)

1491:

Kunze, R. A. An operator theoretic approach to generalized Fourer transforms. Ann. of Math. (2) **69** (1959), 1–14.

Let G be a locally compact abelian group and let f be a measurable function on G. The operator L_f' on $L_2(G)$ is defined as the operator whose domain consists of all g in $L_2(G)$ such that $\int |f(x-y) g(y)| dy$ is locally integrable and $\int f(x-y) g(y) dy$ is in $L_2(G)$ as functions of x, and whose value at g is the latter function. If L_f' is densely defined and has a closure L_f, then L_f is unitarily equivalent via the Plancherel transform to multiplication by a measurable function F on the character group of G. The author calls F the generalized Fourier transform of f, and proceeds to explore this notion. In some ways this notion is well-behaved; for example, anything in $L_p(G)$ for $1 \leq p \leq 2$ has the appropriate generalized Fourier transform, and so does the Heaviside function on the line. On the other hand, this notion is strange in some ways; for example it is not one-to-one (the generalized Fourier transform of any polynomial on the line is 0), and the author does not believe that the set of functions having a generalized Fourier transform is linear.

W. F. Stinespring (Princeton, N.J.)

1492:

Rudin, Walter. Weak almost periodic functions and Fourier-Stieltjes transforms. Duke Math. J. **26** (1959), 215–220.

Let G be a locally compact abelian group, 0 its unit, and $C(G)$ the Banach space of all complex-valued bounded continuous functions on G. W. F. Eberlein has defined a class of functions weak almost periodic (w.a.p.) on G, which is a subclass of $C(G)$, and proved that all Fourier-Stieltjes transforms of bounded regular complex-valued Borel measures on the character group of G and the uniform limits of such functions are w.a.p. [Trans. Amer. Math. Soc. **67** (1949), 217–240; MR **12**, 112]. Eberlein raised the question to the author whether all w.a.p. functions have this property. The author gives a negative answer by proving the following theorem: If (non-compact) G contains a closed discrete subgroup H which is not of bounded order (i.e., there exists no integer $n > 0$ such that $nH = 0$), then there are w.a.p. functions on G which are not uniform limits of Fourier-Stieltjes transforms.

H. Umegaki (Tokyo)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See also 1372.

1493:

Wintner, Aurel. On Heaviside's and Mittag-Leffler's generalizations of the exponential function, the symmetric

stable distributions of Cauchy-Lévy, and a property of the Γ-function. J. Math. Pures Appl. (9) **38** (1959), 165–182.

A collection of statements related to, and typified by, the following: For $0<\lambda<1$, the Mittag-Leffler function

$$E_\lambda(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\lambda n)}$$

can be represented for all z by an integral $\int_0^\infty e^{zt}d\varphi_\lambda(t)$ with a certain $\varphi_\lambda(t)$, for which $d\varphi_\lambda(t)\geqq 0$.

S. Bochner (Princeton, N.J.)

1494:

Zaidman, Samuel. The Laplace-Stieltjes transform of vector-valued functions. Duke Math. J. **26** (1959), 183–188.

Let X be a Banach space. A vector-valued function $\alpha(t)$ on $[0, R]$ to X is said to be of strong bounded variation (in symbols $\alpha\in V^R$) if there exists a constant M such that $\sup \sum_1^n \|\alpha(t_{i+1})-\alpha(t_i)\|\leqq M$ for all partitions $0=t_1<t_2<\cdots<t_n=R$. It is said to be of bounded variation in the Gelfand sense (in symbols $\alpha\in V_G{}^R$) if the set of elements $\sum_1^n \varepsilon_i[\alpha(t_{i+1})-\alpha(t_i)]$, for all partitions $0=t_1<t_2<\cdots<t_n=R$ and all choices $\varepsilon_i=\pm 1$, is conditionally compact in X. E. Hille [see Hille and Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ., Providence, R. I., 1957; MR **19**, 664] has developed a theory of the Laplace transform $f(s)=\lim_R \int_0^R \exp(-st)d\alpha(t)$ when $\alpha\in V^R$ for each $R>0$. In this paper the author develops a parallel theory for functions α in $V_G{}^R$ for all $R>0$. He also shows by examples that neither of these two theories implies the other. *R. S. Phillips* (Los Angeles, Calif.)

1495:

Zaidman, Samuel. On the representation of vector-valued functions by Laplace transforms. Duke Math. J. **26** (1959), 189–191.

The following theorem is proved. In order that a vector-valued function $f(s)$, defined for $s>0$ with values in a Banach space X, may be represented as a Laplace transform, $f(s)=\int_0^\infty e^{-st}\varphi(t)dt$, where $\varphi(t)\in B^\infty[(0,\infty); X]$, and, a null set excepted, have a conditionally compact range, it is necessary and sufficient that (1) the strong derivatives $f^{(k)}(s)$ exist for $k=1, 2, \cdots$ and $s>0$, and (2) the set of elements $s^{k+1}/k!f^{(k)}(s)$, $s>0$, $k=0, 1, 2, \cdots$ is conditionally compact in X. This result generalizes a theorem of Widder [*The Laplace transform*, Princeton Univ. Press, Princeton, N. J., 1941; MR **3**, 232] and is closely related to the work of I. Miyadera [Tôhoku Math. J. (2) **8** (1956), 170–180; MR **18**, 748].

R. S. Phillips (Los Angeles, Calif.)

1496:

Zaidman, Samuel. Sur la représentation des fonctions vectorielles par des intégrales de Laplace-Stieltjes. C. R. Acad. Sci. Paris **247** (1958), 905–907.

This paper is concerned with the representations of vector-valued functions $f(s)$ as Laplace-Stieltjes integrals, that is as $\int_0^\infty \exp(-st)d\alpha$ where $\alpha(t)$ is a normalized function of weak bounded variation on $[0,\infty)$. The function $f(s)$ is said to satisfy condition (A) if (1) $f(s)$ has strong derivatives of all orders for each $s>0$, and (2) the set of elements

$$\sum_1^n \varepsilon_{i,k} \int_{t_i}^{t_{i+1}} L_{k,u}(f(\cdot))du,$$

$n=1,2,\cdots$, $k=1,2,\cdots$, $\varepsilon_{i,k}=\pm 1$, $0<t_1<\cdots<t_{n+1}<\infty$, is bounded; here $L_{k,u}$ is the Widder operator. The author shows that (A) is necessary for the existence of such a representation, and for X weakly sequentially complete that (A) is also sufficient. In case X is an adjoint space, then (A) is sufficient for a representation in which $\alpha(t)$ is normalized in the weak topology.

R. S. Phillips (Los Angeles, Calif.)

1497:

Zaidman, Samuel. Représentation des fonctions vectorielles par des intégrales de Laplace-Stieltjes et compacité faible. C. R. Acad. Sci. Paris **248** (1959), 1915–1917.

In this note the author continues the investigation described in the previous review with somewhat different classes of functions. In this case $\alpha(t)$ is such that the set $\sum_1^n \varepsilon_i[\alpha(t_{i+1})-\alpha(t_i)]$, for all partitions $0<t_1<\cdots<t_{n+1}<\infty$ and all $\varepsilon_i=\pm 1$, is conditionally compact; and the condition (A) is replaced by (A′) which again requires $f(s)$ to have strong derivatives of all orders on $s>0$ and in addition requires the set described in (2) of the previous review to be conditionally weakly compact. The author's principal result is that a necessary and sufficient condition that $f(s)$ be represented as a Laplace-Stieltjes transform with an $\alpha(t)$ of the above type is that $f(s)$ satisfy condition (A′). *R. S. Phillips* (Los Angeles, Calif.)

1498:

Saksena, K. M. Some theorems concerning a generalized Laplace transform. Collect. Math. **10** (1958), 3–19.

A large number of identities involving various special integral transforms are obtained which are analogues of the following result. If

$$f(s) = \int_0^\infty e^{-st}\Phi(t)dt$$

and if

$$\Psi(u) = \int_0^\infty J_0(2\sqrt{(ut)})\Phi(t)dt,$$

then (formally)

$$s^{-1}f(s^{-1}) = \int_0^\infty e^{-su}\Psi(u)du.$$

I. I. Hirschman, Jr. (St. Louis, Mo.)

1499a:

Fung, Kang. Generalized Mellin transforms. I. Acta Math. Sinica **7** (1957), 242–267. (Chinese. English summary)

1499b:

Fung, Kang. Generalized Mellin transforms. I. Sci. Sinica **7** (1958), 582–605.

{#1499b est la traduction du #1499a.}

L'auteur étudie les distributions sur une demi-droite ouverte ("P-distributions") et les formes linéaires sur l'espace des fonctions de type exponentiel à décroissance rapide sur l'axe réel ("Q-distributions"). Il définit le transformée de Mellin d'une P-distribution comme une Q-distribution. Enfin il introduit les deux convolutions multiplicatives des P-distributions et indique des conditions suffisantes (portant sur la croissance des P-distributions données) pour qu'elles soient définies.

J. P. Kahane (Montpellier)

1500:

Arya, Suresh Chandra. Abelian theorem for a generalised Stieltjes transform. Boll. Un. Mat. Ital. (3) **13** (1958), 497–504. (Italian summary)

A familiar Abelian theorem for the Stieltjes transform asserts that if $f(s) = \int_0^\infty (s+t)^{-1} d\alpha(t)$ and if $0<\gamma\leqq 1$ then for any constant A

$$\limsup_{\substack{s\to 0^+ \\ s\to\infty}} |S^\gamma f(S) - A| \leqq \limsup_{\substack{t\to 0^+ \\ t\to\infty}} |\alpha(t) C t^{\gamma-1} - A|,$$

where $C=\Gamma(\gamma)\Gamma(2-\gamma)$; see D. V. Widder, *The Laplace transform* [Princeton Univ. Press, Princeton, N. J., 1941; MR **3**, 232]. In this paper similar theorems are proved with $(s+t)^{-1}$ replaced by the more general kernel

$$\frac{1}{s} F\left[\begin{matrix} 2m+1 \\ m-k+3/2 \end{matrix};\ -\frac{t}{s}\right].$$

I. I. Hirschman, Jr. (St. Louis, Mo.)

1501:

Arya, Suresh Chandra. A real inversion theorem for a generalised Stieltjes transform. Collect. Math. **10** (1958), 69–80.

A number of operational inversion formulas are obtained for integral transforms such as

$$\psi(s) = (2/\pi)^{1/2} \int_0^\infty K_\nu(st)(st)^{1/2}\phi(t)dt,$$

and so on. The demonstrations are vague and unrigorous.

I. I. Hirschman, Jr. (St. Louis, Mo.)

1502:

Stanković, B. Sur les invariants de la transformation intégrale de S. C. Meijer. Acad. Serbe Sci. Publ. Inst. Math. **12** (1958), 53–72.

The author calls the solutions f of

$$(1) \qquad \lambda \int_0^\infty K_\nu(zt)(zt)^{1/2} f(t)dt = f(z)$$

the invariants of the Meijer transform

$$(2) \qquad \int_0^\infty K_\nu(zt)(zt)^{1/2} f(t)dt = F(z),$$

where K_ν is the modified Bessel function of the second kind and $0<\nu<1$. He proves that for each real $\lambda\neq 0$ the functions

$$(3) \qquad f(z) = C[z^{\alpha-1} + \lambda z^{-\alpha} Q(\alpha)]$$

are invariants of (2), where $2\sin\alpha\pi = \pi^2\lambda^2 - 2\cos\nu\pi$ $(\nu-\frac{1}{2} < \operatorname{Re}\alpha < \frac{3}{2}-\nu)$ and $Q(\alpha) = \int_0^\infty K_\nu(t)t^{\frac{1}{2}-\alpha-1}dt$. (For $\lambda = \pm\cos\frac{1}{2}\nu\pi$ a different form of f must be used in (3).) The bulk of the paper is devoted to showing that the functions (3) are the only invariants of (2) which satisfy certain regularity conditions. {Reviewer's note: For $\nu=\frac{1}{2}$ the transform (2) reduces to the Laplace transform. It is known [see D. V. Widder, *The Laplace transform*, Princeton Univ. Press, Princeton, N. J., 1946; MR **3**, 232; p. 390] that in this case the functions (3) are the only invariants of (2) even without regularity conditions.} The proofs consist of straightforward computation using properties of K_ν and some operational calculus.

R. R. Goldberg (Evanston, Ill.)

1503:

Stanković, Bogoljub. Sur les invariants d'un cas spécial de la transformation de S. C. Meijer. Univ. Beogradu. Godišnjak Filozof. Fak. Novom Sadu **3** (1958), 253–264. (Serbo-Croatian. French summary)

Dans une note précédente [analysée ci-dessus #1502] l'auteur a examiné la transformation intégrale dite de S. C. Meijer:

$$\int_0^\infty K_\nu(zt)\sqrt{(zt)} f(t)dt = F(z) \quad (0<\nu<1).$$

Dans cette note l'auteur a complété ses investigations en montrant que les mêmes théorèmes principaux sont aussi valables dans le cas spécial $\nu=0$. La méthode de démonstration est un peu différente de celle de la note citée, à cause du spécial comportement de la fonction $K_0(z)$ au voisinage de $z=0$.

M. Tomić (Belgrade)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 1450.

1504:

Case, K. M. On Wiener-Hopf equations. Ann. Physics **2** (1957), 384–405.

One stage in the Wiener-Hopf method for finding the solution of a class of integral equations involves the splitting of a complex function, analytic in an infinite strip, into two factors. One factor is analytic in one half of the complex plane, one is analytic in the other, and there must be a strip in which both factors are regular.

This factorization is simple in theory, but because it involves infinite line integrals of the Cauchy type it is not often possible to obtain accurate knowledge of the behaviour of the separate factors without a great deal of work. A method of approximate factorization has previously been described by Koiter [Nederl. Akad. Wetensch. Proc. Ser. B **57** (1954), 558–579; MR **17**, 498]. In this paper the author circumvents this problem by transforming the original integral equation into a non-linear one which is often very suitable for obtaining high accuracy approximations to the solution.

V. M. Papadopoulos (Providence, R.I.)

1505:

★Noble, B. Methods based on the Wiener-Hopf technique for the solution of partial differential equations. International Series of Monographs on Pure and Applied Mathematics. Vol. 7. Pergamon Press, New York-London-Paris-Los Angeles, 1958. x+246 pp. $10.00.

The title of this monograph accurately describes its contents. Originally devised to solve an integral equation of the type

$$f(x) = \int_0^\infty f(y)K(x-y)dy$$

for given $K(x)$ in $x>0$, the Wiener-Hopf technique requires a knowledge of certain integral transforms and their inversion, complex integration and some of the properties of complex functions. The theory involved is quite elementary so that the author is concerned, in the

main, with the description of problems solved by this method, and with the subsidiary techniques which have been found useful.

The type of physical problem for which the Wiener-Hopf technique leads to an exact mathematical solution is exemplified by the diffraction of harmonic waves by a half-plane, or by the radiation of sound waves by a semi-infinite pipe. Thus we expect to apply the method to certain "two-part" boundary value problems, for which the boundary conditions differ on the two halves of an infinite surface of constant co-ordinate. The use of Green's functions in this type of problem leads to the definition of the field in terms of a surface integral involving the values of the field and its normal derivative on the boundary, and this leads to an integral equation of the type given above.

The early part of the book shows the way to avoid even thinking of this integral equation. {D. S. Jones is given the credit for this approach, although Carrier used the same method a couple of years earlier.} The use of integral transforms (only Fourier or Laplace, and Mellin transforms are known to be of value) in these problems leads to a problem in the theory of complex functions. We have to consider the splitting of a function $K(s)$ which is regular in an infinite vertical strip of the complex s-plane into two factors with a common strip of regularity, one regular in each half (left-hand or right-hand) of the s-plane. In general these factors are complex line integrals. The author gives many cases in which this factorization is simple, and he describes Koiter's method of approximate factorization for more difficult functions. {A different approach to this question has recently been described by Case [see the preceding review].}

He also describes the application of the Wiener-Hopf method to more complicated problems such as the diffraction effect of a thick slab or of a step discontinuity in a pipe. Such problems lead to an infinite set of linear equations with an infinite set of unknowns which may be solved only in the crudest manner. {It is the reviewer's opinion that the same equations, or better still an integral equation of Fredholm type, may be reached more rapidly by less sophisticated methods; the solution need then be no more crude.}

The author also discusses the close relation between the complex function theory used in the Wiener-Hopf method and that used in the solution of the Riemann-Hilbert problem. He examines the possibility of finding, by setting some problem of Riemann-Hilbert type, results in some "two part" boundary value problems when other integral transforms (e.g., Lebedev-Kontorovich transforms) are appropriate.

This is a very readable book which has been written to help the applied mathematician. There is abundance of detailed examples in which the author summarizes and comments on about one hundred original papers. He has kept the amount of rigorous argument low; this has enabled him to discipline the large amount of material that he presents.

V. M. Papadopoulos (Providence, R.I.)

1506:

Gohberg, I. C.; and Kreĭn, M. G. Systems of integral equations on the half-line with kernels depending on the difference of the arguments. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 2 (80), 3–72. (Russian)

The article is devoted to the system

$$\text{(A)}\quad \chi(t)-\int_0^\infty k(t-s)\chi(s)ds = f(t) \quad (0 \leq t < \infty),$$

where $k(t)$ is a given n-by-n matrix with elements in $L_1(-\infty, \infty)$; the elements of the given n-vector $f(t)$ and the unknown n-vector $\chi(t)$ may lie in $L_1(0, \infty)$, though a number of other spaces are also considered. A brief survey of the theory of (A), with special reference to operator theory, was appended to a previous article of the authors [same Uspehi **12** (1957), no. 2 (74), 43–118; MR **20** #3459]. One of the authors (Kreĭn) has given a very extensive account of (A) in the scalar case $n=1$ [see the paper reviewed below]; the reviewer found it advisable to read Kreĭn's paper first, although it actually appeared later. The paper under review commences, apart from introductory and ancillary matter, with a study of the homogeneous equation

$$\text{(B)}\quad \phi(t)-\int_0^\infty k(t-s)\phi(s)ds = 0$$

and its transpose

$$\text{(C)}\quad \psi(t)-\int_0^\infty k'(s-t)\psi(s)ds = 0.$$

If $\det(I-\mathscr{K}(\lambda))\neq 0$, $-\infty<\lambda<\infty$, where $I=$unit matrix and $\mathscr{K}(\lambda)=\int_{-\infty}^{\infty} k(t)e^{i\lambda t}dt$, then (B) and (C) have each only a finite number of linearly independent solutions in $L_1(0, \infty)$, which are the same as the solutions in certain other spaces. A more incisive result gives the difference between the two numbers of solutions as

$$(2\pi)^{-1}[\arg\{\det(I-\mathscr{K}(\lambda))\}]_{-\infty}^{\infty},$$

reckoned along the real axis; the authors are able to reduce this to the case $n=1$, considered by Kreĭn.

The authors next develop the connection between the solubility of the homogeneous equations and the "factorisation" of $I-\mathscr{K}(\lambda)$. There is first the "Hilbert problem" of finding matrices $\mathfrak{F}_\pm(\lambda)$ such that $(I-\mathscr{K}(\lambda))\mathfrak{F}_+(\lambda)=\mathfrak{F}_-(\lambda)$ $(-\infty<\lambda<\infty)$, where $\mathfrak{F}_+(\lambda)$, $\mathfrak{F}_-(\lambda)$ are holomorphic in the upper and lower half-planes, respectively, including the boundaries. In the standardised problem $\det \mathfrak{F}_+(\lambda)$ has its only zeros in the upper half-plane at $\lambda=i$, while $\det \mathfrak{F}_-(\lambda)\neq 0$ for $\operatorname{Im}\lambda\leq 0$. It turns out that $\mathfrak{F}_+(\lambda)$, $\mathfrak{F}_-(\lambda)$ belong to certain rings $\mathfrak{R}^+_{(n\times n)}$, $\mathfrak{R}^-_{(n\times n)}$, where for example $\mathfrak{R}^+_{(n\times n)}$ is the set of matrix functions of the form $C+\int_0^\infty F(t)e^{i\lambda t}dt$, and the elements of $F(t)$ are in $L_1(0, \infty)$. Secondly, there is the problem of finding a matrix analogue of the factorisation $(I-\mathscr{K}(\lambda))^{-1}=\mathfrak{G}_+(\lambda)\mathfrak{G}_-(\lambda)$, which plays a fundamental role in the scalar or Wiener-Hopf case. The authors set up, for the essentially more complicated matrix case, a "left standard factorisation" $(I-\mathscr{K}(\lambda))^{-1}=\mathfrak{R}_+(\lambda)\mathfrak{D}(\lambda)\mathfrak{R}_-(\lambda)$, where $\mathfrak{R}_+(\lambda)\in\mathfrak{R}^+_{(n\times n)}$, $\det\mathfrak{R}_+(\lambda)\neq 0$ in the closed upper half-plane and analogously for $\mathfrak{R}_-(\lambda)$, while $\mathfrak{D}(\lambda)$ is a diagonal matrix with diagonal elements of the form $[(\lambda-i)/(\lambda+i)]^{\kappa_j}$, $j=1, \cdots, n$. The "partial indices" κ_j appear also in the Hilbert problem and, what is newer, clarify the solution of the homogeneous equations. If, for example, $\kappa_j>0$, there corresponds a chain of solutions $\phi_j(t)$ of (B) related by $\phi_j(t)=\int_0^t \phi_{j+1}(s)ds$ $(j=0, \cdots, \kappa_j-1)$. For a negative κ_j one gets a chain of solutions of (C).

The topics of factorisation and partial indices possess an interest independently of (A), (B), (C). The authors

pursue this interest in regard to Hermitian and triangular matrices; they also investigate perturbation properties and stability of the partial indices.

The final sections deal with (i) the solutions of (A) in the form $\chi(t)=f(t)+\int_0^t \gamma(t,s)f(s)ds$, where $\gamma(t,s)$ is related to the factorisation of $(I-\mathscr{K}(\lambda))^{-1}$, (ii) the spectrum of the operator occurring in (A), and (iii) the discrete analogue of (A), for which the factorisation takes place on the unit circle. *F. V. Atkinson* (Canberra City)

1507:

Krein, M. G. Integral equations on the half-line with a kernel depending on the difference of the arguments. Uspehi Mat. Nauk **13** (1958), no. 5 (83), 3–120. (Russian)

The primary aim of this paper is a detailed study of the integral equation

$$\text{(A)}\qquad \chi(t)-\int_0^\infty k(t-s)\chi(s)ds = f(t) \quad (0 \leq t < \infty);$$

the homogeneous equation

$$\text{(B)}\qquad \chi(t)-\int_0^\infty k(t-s)\chi(s)ds = 0 \quad (0 \leq t < \infty);$$

and the associated homogeneous equation

$$\text{(C)}\qquad \chi(t)-\int_0^\infty k(s-t)\chi(s)ds = 0.$$

In addition, there is a fairly complete study of the discrete analog of (A), namely, the infinite system of equations

$$(1)\qquad \sum_{k=0}^{\infty} a_{j-k}\xi_k = \eta_j \quad (j=0,1,2,\cdots),$$

where the coefficients satisfy $\sum_{j=-\infty}^{\infty}|a_j|<\infty$. Under special restrictions, consideration is also given to the integral equation of the first kind

$$(2)\qquad \int_0^\infty k(t-s)\chi(s)ds = f(t) \quad (0 \leq t < \infty).$$

Fully eighteen pages at the close of the paper are devoted to examples illustrating methods and techniques for carrying out the detailed steps in solving specific equations of types (A) and (2).

In the consideration of (A), (B), (C) it is assumed that the kernel $k(t)\in L_1(-\infty,\infty)$. The complex-valued functions $f(t)$ and $\chi(t)$ are considered as belonging to the Banach space E, which, in the present paper, may be any one of the spaces $L_+^{(p)}$ $(p\geq 1)$, C_+^0, C_+, M_+^u, M_+^c, M_+. The space $L_+^{(p)}$ is the space of all measureable functions $f(t)$ $(0\leq t<\infty)$ with $\int_0^\infty |f(t)|^p dt<\infty$, and norm $\|f\|_p=(\int_0^\infty |f(t)|^p dt)^{1/p}$; M_+ is the space of all bounded functions $f(t)$ $(0\leq t<\infty)$, with norm $\|f\|_M=\sup_{0<t<\infty}|f(t)|$. The spaces M_+^c, M_+^u are subspaces of M_+, the first consisting of all continuous functions, the second of all uniformly continuous functions. Finally C_+ is the space of all continuous functions $f(t)$ $(0\leq t<\infty)$ for which $f(\infty)=\lim_{t\to\infty}f(t)$ exists, and C_+^0 is its subspace for which $f(\infty)=0$. Thus, $C_+^0\subset C_+\subset M_+^u\subset M_+^c\subset M_+$.

The author arrives at three principal theorems, which we state below.

Theorem 1: Let $k(t)\in L_1(-\infty,\infty)$. Then for arbitrary $f\in E$, equation (A) will have one and only one solution $\chi\in E$ if and only if

$$(*)\quad 1-K(\lambda)\neq 0 \quad (-\infty<\lambda<\infty), \quad \nu=-\operatorname{ind}(1-K)=0,$$

where $K(\lambda)$ is the Fourier transform of $k(t)$ and $\nu=-(2\pi)^{-1}\int_{-\infty}^{\infty} d_\lambda \arg(1-K(\lambda))$ is the index of (A). The solution will be given by

$$(3)\qquad \chi(t) = f(t)+\int_0^\infty \gamma(t,s)f(s)ds,$$

wherein the resolvent kernel has the formulation

$$(4)\ \gamma(t,s) = \gamma(t-s,0)+\gamma(0,s-t)+\int_0^\infty \gamma(t-r,0)\gamma(0,s-r)dr$$

$$(\gamma(t,0) = \gamma(0,t) = 0 \quad \text{for} \quad t<0).$$

The functions $\gamma(t,0)$, $\gamma(0,t)$ belong to $L_1(0,\infty)$ and are in turn determined uniquely from the relations

$$(5)\qquad (1-K(\lambda))^{-1} = G_+(\lambda)G_-(\lambda) \quad (-\infty<\lambda<\infty),$$

$$(6)\qquad G_+(\lambda) = 1+\int_0^\infty \gamma(t,0)\exp(i\lambda t)dt,$$

$$G_-(\lambda) = 1+\int_0^\infty \gamma(0,t)\exp(-i\lambda t)dt,$$

$$(7)\ G_+(\lambda)\neq 0 \quad (\operatorname{Im}\lambda\geq 0), \qquad G_-(\lambda)\neq 0 \quad (\operatorname{Im}\lambda\leq 0).$$

The factorization (5) with the factors normalized so that $G_\pm(\infty)=1$ is unique whenever (*) is satisfied and conversely.

Theorem 2: If $k(t)\in L_1(-\infty,\infty)$ and $1-K(\lambda)\neq 0$ $(-\infty<\lambda<\infty)$, then (B) will have non-trivial solutions in E if and only if $\nu>0$. (The same statement applies to (C) but with $\nu<0$.) These solutions will be the same for all the component spaces of E and, in fact, will have a basis consisting of functions $\varphi_j(t)$ $(j=0,1,\cdots,\nu-1)$, absolutely continuous, with derivatives belonging to $L_1(0,\infty)$, tending to zero as $t\to\infty$, and related, in sequence, by

$$(8)\quad \varphi_{k+1}(t) = \frac{d\varphi_k}{dt}, \qquad \varphi_k(0)=0 \quad (k=0,1,2,\cdots,\nu-2),$$

$$\varphi_{\nu-1}(0)\neq 0.$$

As a by-product of the proof of Theorem 2, the author furnishes an analytical procedure for determining the $\varphi_j(t)$.

The applicable theorem for (A) when the index $\nu\neq 0$ is Theorem 3: If $k(t)\in L_1(-\infty,\infty)$ and $1-K(\lambda)\neq 0$, then for $\nu>0$ and $f\in E$, equation (A) will possess an infinite set of solutions $\chi\in E$. However, if $\nu<0$, then (A) will have either no solution or a single unique solution, the latter obtaining when $\int_0^\infty f(t)\psi_j(t)dt=0$ $(j=0,1,2,\cdots,|\nu|-1)$, where the $\psi_j(t)$ represent a basis for the solutions of (C).

The paper contains a long introductory section, followed by four principal chapters, a page or two of supplementary remarks and a very complete bibliography. The introduction traces the history of equations (A), (B), (2) and the particular restrictions employed in their treatment, citing in detail the work of N. Wiener, E. Hopf, E. Reissner, V. Fok and I. Rapoport. The problem of factorization of $(1-K(\lambda))^{-1}$, as represented by (5) of Theorem 1 and due originally to Wiener and Hopf, is treated in Chapter I, consideration being given successively to factorization on the real line (needed in Theorem 1), on a strip, and on a circle (used in solving (1)); in addition, there is an independent section on factoring entire functions of the form $1+\int_{a_-}^{a_+} g(t)\exp(i\lambda t)dt$ on the real line, where $a_\pm$ are finite and $g\in L_1(a_-,a_+)$. The proofs of Theorems 1, 2, 3 and the solution of systems of type (1) are discussed in Chapters II, III, a by-product of the

analysis being a study of the spectrum of the operator K in E where $K\varphi=\int_0^\infty k(t-s)\varphi(s)ds$. Chapter IV studies (A), (B) for the Wiener-Hopf type case where $k(t)$ is assumed to satisfy a condition of the form $k(t)\exp(h(t))\in L_1(-\infty,\infty)$ $(h>0)$. New material here is reflected in a series of theorems concerned with obtaining asymptotic representations of the solutions of (B) in terms of exponentials with polynomial coefficients. This is followed by the aforementioned section on detailed examples for special cases.

J. F. Heyda (Cincinnati, Ohio)

1508:

Heins, A. E.; and MacCamy, R. C. A function-theoretic solution of certain integral equations. I. Quart. J. Math. Oxford Ser. (2) **9** (1958), 132–143.

The authors present an extension of a method originated by Carleman (and quite distinct from the Wiener-Hopf method) for solving certain integral equations of the form

$$(1)\qquad \int_0^\infty K(|x-\xi|)f(\xi)d\xi = g(x)\quad (x>0).$$

If $K(w)$ is of the form $P(w)\log w+Q(w)$, where $P(w)$ and $Q(w)$ are even and entire, and if $g(x)$ can be continued to the entire complex plane except for isolated singularities, it is shown that the (multiple-valued) function

$$(2)\qquad F(z) = \int_0^\infty K(\xi-z)f(\xi)d\xi$$

satisfies a linear functional relationship involving $g(z)$. Under suitable hypotheses concerning the asymptotic behavior of $K(w)$ it is possible to determine $F(z)$ from this functional relationship, and then (1) and (2) furnish a new integral equation for $f(\xi)$, similar to (1) but of Volterra type, which can be solved by Laplace transform. The method is illustrated by a detailed treatment of an acoustical diffraction problem.

Bernard Epstein (Philadelphia, Pa.)

1509:

Neuberger, J. W. Continuous products and nonlinear integral equations. Pacific J. Math. **8** (1958), 529–549.

The author considers integral equations of the form

$$(1)\qquad Y(t) = A+\int_c^t dF\cdot Y.$$

Here c is a point contained in the interval $[a,b]$ of the real axis while t varies in a small enough subinterval Q of $[a,b]$ containing c. The unknown $Y=Y(t)$ is a function from Q to a space S which is an additive Abelian group with elements $x, y, \cdots$ having a non-negative norm $\|x\|$ with the following properties: $\|x\|=0$ only for the zero element of S, $\|-x\|=\|x\|$, $\|x+y\|\leq\|x\|+\|y\|$. With respect to this norm S is supposed to be complete. A is a given element of S, and F is a function from $[a,b]$ to the space B of all continuous (not necessarily linear) transformations from S into S. The meaning of the integral in (1) is (under certain assumptions about F) made clear in section 5 of the paper. The main result (again with suitable assumptions about F) is that (1) has one and only one continuous solution and that this solution may be written as a continuous product $\prod_c^t(T,A)$ where $T(t',t'')$ is a function from the square $[a,b]\times[a,b]$ to B. The precise definition is given in section 2 for a complete metric space S, i.e., a more general space S than the one defined above. Roughly speaking the continuous product is defined as a limit element in S of $[\prod_{i=1}^n T(t_{i+1},t_i)]A$ where $\{t_i\}$ is a subdivision of $[c,t]$. In case S is normed linear and metric the solution of (1) in terms of a continuous product has been given by J. S. MacNerney [Ann. of Math. (2) **61** (1955), 354–367; MR **16**, 716].

In the remaining part of the paper the author obtains bounds for errors in computing continuous products by various approximation procedures and treats some examples.

E. H. Rothe (Ann Arbor, Mich.)

FUNCTIONAL ANALYSIS

See also 1379, 1455, 1469, 1494, 1495, 1496, 1497, 1499a–b.

1510:

Cristescu, Romulus. À propos des théorèmes ergodiques individuels de H. Nakano. An. Univ. "C. I. Parhon" București. Ser. Şti. Nat. **7** (1958), no. 20, 23–25. (Romanian. Russian and French summaries)

The author generalizes from normed to seminormed linear σ-lattices an ergodic theorem proved by H. Nakano [Ann. of Math. (2) **49** (1948), 538–556; MR **10**, 550].

M. M. Day (Urbana, Ill.)

1511:

Silverman, R. J.; and Yen, Ti. The Hahn-Banach theorem and the least upper bound property. Trans. Amer. Math. Soc. **90** (1959), 523–526.

Let V be a partially ordered linear space. It was shown by M. M. Day [Notes on ordered linear spaces, Univ. of Illinois, 1950, unpublished] that the Hahn-Banach extension theorem about linear functions dominated by a positive homogeneous subadditive function is also valid in case the values of functions are taken from V, if every bounded set of elements in V has a least upper bound. The authors prove that the converse is true if we assume that every line in V intersects the positive cone in a closed segment.

I. G. Amemiya (Kingston, Ont.)

1512:

Singer, I. The problem of homeomorphism of separable infinite-dimensional Banach spaces. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **12** (1958), no. 4 (27), 5–24. (Romanian)

This is an enthusiastic exposition designed to arouse interest among Romanian mathematicians in a problem still unsolved almost thirty years after it was stated in Banach's book *Théorie des opérations lineaires*. The bibliography reaches from 1926 to late 1958 and from Long and Klee through Mazur and Nikolsky to Kadec, whose papers [Dokl. Akad. Nauk SSSR **92** (1953), 465–468; **122** (1958), 13–16; Uspehi Mat. Nauk (N.S.) **10** (1955), no. 4 (66), 137–141; MR **15**, 535; **20** #5422; **17**, 511] contain the biggest steps toward solving this problem.

M. M. Day (Urbana, Ill.)

1513:

Pełczyński, A. On the isomorphism of the spaces m and M. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 695–696.

Let m, M, l and L be the Banach spaces of bounded sequences, of bounded measurable functions on $[0, 1]$, of summable sequences and of summable functions on $[0, 1]$. The author indicates a proof that m and M are (topologically) isomorphic. Therefore, in spite of the fact that l has smaller linear dimension than L, the dual spaces of l and L are isomorphic. *L. Nachbin* (Rio de Janeiro)

1514:

Buck, R. C. A complete characterization for extreme functionals. Bull. Amer. Math. Soc. **65** (1959), 130–133.

Let E be a normed real linear space and S the unit sphere of the dual space E^*. The author characterizes the extreme points of S as follows. If $L \in S$, let p^* be the largest seminorm on E satisfying $p^*(x) \leqq \|x\| - L(x)$ and $V_L = \{x \in E;\ p^*(x) = 0\}$. Then L is extreme in S if and only if $V_L = E$. A more explicit description of V_L and p^* is given in the following way. For $k > 0$, let

$$F_k = \{x \in E;\ \|x\| - L(x) \leq 1/k\}.$$

Then $V_L = \bigcap (F_k - F_k)$. Let also

$$p_k(x) = \inf\{\|x+u\| - L(x+u);\ u \in F_k\},$$

$$p(x) = \lim_{k\to\infty} p_k(x).$$

Then

$$p^*(x) = \inf\{p(x+z) + p(z);\ z \in E\}.$$

A slightly more general result and an application to approximation theory are indicated.
L. Nachbin (Rio de Janeiro)

1515:

Gagliardo, Emilio. Interpolation d'espaces de Banach et applications. C. R. Acad. Sci. Paris **248** (1959), 1912–1914.

Let A, B be Banach spaces with A embedded linearly and continuously in B and A dense in B. For u in B the symbol $\Gamma(u)$ denotes a curve, $t \to v(t)$, $0 \leqq t < 1$, in A with the properties: (i) the map $t \to v(t)$ is continuous from $0 \leqq t < 1$ to A; (ii) $v(0) = 0$; and (iii) considered as a mapping into B the function v is of bounded variation on every closed interval $0 \leqq t \leqq s$ with $s < 1$, and $v(t) \to u$ as $t \to 1$. For each u in B and each number θ with $0 < \theta < 1$ the norm $|u|_{A^{1-\theta}B^\theta}$ is defined by the expression

$$\inf_{\Gamma(u)} \lim_{s\to 1} \lim_{\delta\to 0} \sum_{i=1}^{n} |v(\tau_i)|^{(1-\theta)/\theta} |v(t_i) - v(t_{i-1})|_B,$$

where δ is the norm of the participating $0 = t_0 < t_1 < \cdots < t_n = s$, and $t_{i-1} \leqq \tau_i \leqq t_i$. The set of those u in B for which this norm is finite form a Banach space $A^{1-\theta}B^\theta$ under this norm such that $A \subset A^{1-\theta}B^\theta \subset B$ both algebraically and topologically. If $A = L_\infty(K)$, $B = L_1(K)$ where K is a compact set in Euclidean space, then with $p = 1/\theta$ the space $A^{1-\theta}B^\theta$ is $L_p(K)$. Other examples and applications are given. *N. Dunford* (Brooklyn, N.Y.)

1516:

Martirosyan, R. M. A biorthogonal system. Akad. Nauk Armyan. SSR Dokl. **27** (1958), 3–11. (Russian. Armenian summary)

The author considers the equation

$$(A^* + \lambda w_2 E)(A - \lambda w_1 E)\varphi = 0,$$

where A is a symmetric operator on a Hilbert space H. This is evidently equivalent to (1) $A\varphi_1 - \lambda w_1 \varphi_1 = \varphi_2$, $A^*\varphi_2 + \lambda w_2 \varphi_2 = 0$.

Theorem: If an inner product on $H \times H$ is defined by $(\{\varphi_1, \varphi_2\}, \{\psi_1, \psi_2\}) = w_1(\varphi_1, \psi_1) + w_2(\varphi_2, \psi_2)$ $(w_1, w_2 > 0)$, then the eigenvalues of (1) are all real, and coincide (including multiplicity) with those of (2) $A\psi_2 + \lambda w_2 \psi_2 = -\psi_1$, $A^*\psi_1 - \lambda w_1 \psi_1 = 0$. If these eigenvalues are λ_k and the corresponding solutions of (1) and (2) are $\Phi_k = \{\varphi_1^k, \varphi_2^k\}$, $\Psi_k = \{\psi_1^k, \psi_2^k\}$, then Φ_k and Ψ_k form a biorthogonal system on $H \times H$ in the inner product defined above.
R. R. D. Kemp (Kingston, Ont.)

1517:

Kohls, C. W. Ideals in rings of continuous functions. Fund. Math. **45** (1957), 28–50.

This paper contains a considerable number of results on the ideals (and ideals of ideals and subrings) of the ring $\mathscr{C}(X, R)$ of continuous real-valued functions on a completely regular space X. It is not self-contained, requiring reference to Gillman and Henriksen, Trans. Amer. Math. Soc. **77** (1954), 340–362 [MR **16**, 156]; see also Kohls, same Fund. **45** (1957), 17–27 [MR **20** #7040], and the papers reviewed below. The occurrence of many concepts of contrived appearance, but actually very appropriate nature, makes impossible the quotation of more than a few results in this review. Let $\mathscr{C}_0$ [$\mathscr{C}_\infty$] be the functions that vanish at [and, respectively, in a neighborhood of] infinity. Then $\mathscr{C}_\infty$ lies in no maximal ideal of $\mathscr{C}_0$ (ideals which are not linear subspaces are not excluded from consideration). For a point p in βX let M^p be the class of f in $\mathscr{C}(X)$ such that p is a limit point of the set $\{f = 0\}$. (According to Gelfand and Kolmogoroff, these M^p are the maximal ideals of $\mathscr{C}(X)$.) Let M'^p be the class of those f such that $\{f = 0\}$ intersects each deleted neighborhood of p; and let N^p be all those f which vanish on $\Omega \cap X$ for a suitable neighborhood Ω of p. Suppose N^p is prime. Then M'^p is also prime ($N^p \subset M'^p \subset M^p$) and the prime ideals including M'^p form a chain. *R. Arens* (Los Angeles, Calif.)

1518:

Kohls, Carl W. Prime ideals in rings of continuous functions. Illinois J. Math. **2** (1958), 505–536.

The author has carried out a thorough study of the properties of the set of all prime ideals P of the ring $C = C(X)$ of all continuous real-valued functions on a completely regular space X. In particular, the structure of the residue-class rings C/P and the structure of the partially ordered set of prime ideals P contained in a given maximal ideal of C are studied.

Let βX denote the Stone-Čech compactification of X, and let vX denote the largest subspace of βX over which every $f \in C(X)$ has a continuous real-valued extension. If $p \in \beta X$, let $M^p = \{f \in C(X)$: p is in the closure in βX of $f^{-1}(0)\}$, and let $N^p = \{f \in C(X)$: there is a neighborhood Ω of p such that $f^{-1}(0) \supset \Omega \cap X\}$. It is known that an ideal M of $C(X)$ is maximal if and only if there is a $p \in \beta X$ such that $M = M^p$, and if an ideal P of $C(X)$ is prime, then there is a unique $p \in \beta X$ such that $N^p \subset P \subset M^p$ [see L. Gillman, M. Henriksen and M. Jerison, Proc. Amer. Math. Soc. **5** (1954), 447–455; MR **16**, 607; and Gillman and Henriksen, cited in #1517 above].

If P is a prime ideal of C containing N^p, then C/P is a totally ordered integral domain containing the real field

R, and C/P has infinitely large elements if and only if $p \notin \upsilon X$. Moreover, the prime ideals containing P form a chain.

Let C^* denote the ring of bounded elements of C, and let $(C/P)_t = \{a \in C/P: |a| \text{ is not infinitely large}\}$. There is an order-preserving isomorphism of $(C/P)_t$ onto $C^*/P \cap C^*$.

An ordered set L is called an η_1-set provided that (i) if A and B are countable subsets of L such that $A < B$, then there is a $y \in L$ such that $A < y < B$, and (ii) L has no countable coinitial or cofinal subset. The author shows if D, D' are countably infinite subsets of C/P of order type ω and ω^*, respectively, then there is a $\delta \in C/P$ such that $D < \delta < D'$. Moreover, C/P has a countable cofinal (and coinitial) subset if and only if $p \in \upsilon X$. But C/P may not be an η_1-set; indeed the set of positive elements of C/P may have a countable coinitial subset.

Let P be nonmaximal, and let a be a positive element of M^p/P. The upper ideal Q^a is the intersection of all the prime ideals of C/P containing a, and the lower ideal Q_a is the largest ideal of C/P disjoint from $\{a, a^2, \cdots, a^n, \cdots\}$. The set of upper ideals and the set of lower ideals are disjoint densely ordered subsets of the set of prime ideals of C/P. Each upper ideal is countably generated, but no lower ideal is. There exist uncountably many prime ideals of both types, but not every prime ideal of C/P need be an upper or a lower ideal.

An ideal I of C is a $\mathscr{Z}$-ideal if $f \in I$ and $f^{-1}(0) = g^{-1}(0)$ imply that $g \in I$. If I is a $\mathscr{Z}$-ideal of C containing N^p, then I is prime if and only if the prime ideals of C containing I form a chain.

A totally ordered valuation ring is a totally ordered ring A such that if $a, b \in A$ and $0 \leq a \leq b$, then a is a multiple of b. The author determines when a $\mathscr{Z}$-ideal I of C containing N^p is such that C/I is a totally ordered valuation ring. This latter is equivalent, in particular, to the assumption that the ideals of C containing I form a chain.

The author turns next to βF-points in the sense of an earlier paper [#1517], where it is noted that p is a βF-point if and only if N^p is a prime ideal. A necessary condition that C/N^p be a totally ordered valuation ring is that p be a βF-point. It is not known if this is sufficient, but two sets of sufficient conditions are given which improve results in the paper cited above.

Many pertinent examples are given. {D. Johnson, M. Jerison, and the reviewer have discovered that Example 4.2 is not correct. The set of positive elements of $C(E^{(\omega)}/N^e)$ has no countable coinitial subset.}

M. Henriksen (Lafayette, Ind.)

1519:

Kohls, Carl W. Prime ideals in rings of continuous functions. II. Duke Math. J. **25** (1958), 447–458.

This is a continuation of the investigations in part I [#1518 above]. Familiarity with that paper is necessary for an appreciation of this one. The main theorem here "enables one to find certain non-maximal prime $\mathscr{Z}$-ideals of $\mathscr{C}(X-\{p\})$ whenever p is a G_δ-point."

R. Arens (Los Angeles, Calif.)

1520:

Horne, J. G., Jr. On O_ω-ideals in $C(X)$. Proc. Amer. Math. Soc. **9** (1958), 511–518.

An O_ω-ideal of a semigroup S is an ideal J of S such that $f_1, f_2 \in J$ implies there is an $e \in J$ such that f_1 and f_2 are multiples of e. If S is also a ring, then an O_ω-ideal of S is clearly also a ring ideal. If $S = C(X)$, the ring of all continuous real-valued functions on a (completely regular) space X, then every intersection of prime ideals is an O_ω-ideal. If T is a multiplicative isomorphism of $C(X)$ onto $C(Y)$, then T sends maximal ideals, intersections of prime ideals, and m-closed ideals [for definition, see L. Gillman, M. Henriksen and M. Jerison, Proc. Amer. Math. Soc. **5** (1954), 447–455; MR **16**, 607] into ideals with the corresponding properties. Every ideal of $C(X)$ is principal if and only if every ideal of $C(X)$ is an O_ω-ideal.

An O-ideal is an ideal J such that if $f_1, f_2 \in J$, there is an $e \in J$ such that $f_i = f_i e$ $(i = 1, 2)$. The author obtains a necessary and sufficient condition for an intersection of prime ideals of $C(X)$ to be an O-ideal in terms of a topology on the space of proper prime ideals of $C(X)$.

M. Henriksen (Lafayette, Ind.)

1521:

Warner, Seth. The topology of compact convergence on continuous function spaces. Duke Math. J. **25** (1958), 265–282.

The author makes a fairly complete study of the topological vector space $\mathscr{C}(T)$ of all real-valued continuous functions on a completely regular space T. He discusses the relationship between properties of T and properties of $\mathscr{C}(T)$, such as $\mathscr{C}(T)$ being metrizable, bornological, barreled, complete, quasi-complete, a Fréchet space, separable, symmetric or "infratonnélé", a nuclear space, a Schwartz space, a Montel space, reflexive or semi-reflexive, a (DF) space, a space satisfying Mackey's convergence conditions, etc. Most of the results are final but some questions are only partially answered.

L. Nachbin (Rio de Janeiro)

1522:

Onuchic, Nelson. P-spaces and the Stone-Čech compactification. An. Acad. Brasil. Ci. **30** (1958), 43–45.

Let $C(E, R)$ denote the set of continuous functions on the completely regular space E with values in the real line R. Let $\mathscr{U}_N(E)$, $\mathscr{U}_0(E)$ denote, respectively, the coarsest uniform structure on E making every element of $C(E, R)$ uniformly continuous, and the finest uniformity on E. If E is also a P-space in the sense of L. Gillman and the reviewer [Trans. Amer. Math. Soc. **77** (1954), 340–362; MR **16**, 156] (i.e. if every G_δ in E is open), let $\mathscr{U}_P(E)$ denote the uniformity generated by all sets of the form $\bigcup_{r \in R} f^{-1}(r) \times f^{-1}(r)$, as f ranges over $C(E, R)$.

The author announces the following results. If E is a P-space, then $\mathscr{U}_P(E) = \mathscr{U}_N(E)$ if and only if every open partition of E is countable; $\mathscr{U}_P(E \times E) = \mathscr{U}_P(E) \times \mathscr{U}_P(E)$ if and only if $\mathscr{U}_P(E) = \mathscr{U}_0(E)$. Indeed, this latter holds if every open covering of E has a subcover of power $\leq \aleph_1$.

Every point-wise convergent sequence of elements of $C(E, R)$ has its limit in $C(E, R)$ if and only if E is a P-space. Every separately continuous real-valued function on $E \times E$ is in $C(E, R)$ if and only if E is discrete.

The author also announces some special cases of the following result of I. Glicksberg [Trans. Amer. Math. Soc. **90** (1959), 369–382]. Let βE denote the Stone-Čech compactification of E. In order that there exist a homeomorphism of $\beta(E \times E)$ onto $\beta E \times \beta E$ keeping $E \times E$ pointwise fixed it is necessary and sufficient that every $f \in C(E \times E, R)$ be bounded.

No proofs are given. *M. Henriksen* (Lafayette, Ind.)

1523:

Tamano, Hisahiro. On rings of real valued continuous functions. Proc. Japan Acad. **34** (1958), 361–366.

A number of properties of a completely regular space X are expressed (partially) in terms of the ring $C(X)$ of continuous real-valued functions on X. Some alternate proofs of known theorems are given, and some new results are derived, two of which are given below.

An ideal I of $C(X)$ is called locally finite if there exists a family $\{\phi_\lambda\}_{\lambda\in\Gamma}$ of elements of I such that $0\leq\phi_\lambda\leq 1$ for all $\lambda\in\Gamma$, every point of X has a neighborhood outside of which all but finitely many ϕ_λ's vanish, and $\sum_{\lambda\in\Gamma}\phi_\lambda(x)=1$ for all $x\in X$. The author shows that X is paracompact if and only if every free ideal of $C(X)$ is locally finite, and that X admits a complete uniformity if and only if every maximal free ideal of $C(X)$ is locally finite. (An ideal I of $C(X)$ is called free if for every $x\in X$, there is an $f\in I$ such that $f(x)\neq 0$.) *M. Henriksen* (Lafayette, Ind.)

1524:

Zawadowski, W. Axiomatic characterization of some rings of real functions. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 355–360.

Certain archimedian lattice-ordered rings are represented as subrings of a ring of real-valued functions each of which is defined on an everywhere dense open subset of a compact (Hausdorff) space.

M. Henriksen (Lafayette, Ind.)

1525:

Gagliardo, Emilio. Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili. Rend. Sem. Mat. Univ. Padova **27** (1957), 284–305.

Le problème est essentiellement le suivant: on considère dans R^n l'ouvert $x_n>0$, soit Ω, de frontière Γ $(x_n=0)$. Soit $W_p(\Omega)$ l'espace des fonctions $u\in L_p(\Omega)$ de dérivées distributions $\partial u/\partial x_i$ dans $L_p(\Omega)$. L'espace $\mathcal{D}^1(\overline{\Omega})$ des fonctions une fois continument différentiables à support compact dans $\overline{\Omega}$, est dense dans $W_p(\Omega)$, et on constate facilement que l'application $u\to u(x',0)$ $(x'=(x_1,\cdots,x_{n-1}))$ est continue de l'espace $\mathcal{D}^1(\overline{\Omega})$ muni de la topologie induite par $W_p(\Omega)$ dans, par exemple, l'espace des fonctions localement sommables sur Γ. On peut donc prolonger cette application par continuité, et on a ainsi défini, pour tout $u\in W_p(\Omega)$ une trace $u(x',0)$, localement sommable. Le problème est de caractériser directement l'espace des $u(x',0)$ lorsque u parcourt $W_p(\Omega)$. (Naturellement on passe de là au cas de Ω "généraux", de frontière assez régulière, par cartes locales.) Si $p=2$, la résolution de ce problème est facile par transformation de Fourier; l'A. résout le problème—beaucoup plus difficile—pour p quelconque. Si $p=1$, l'espace des traces est $L^1(\Gamma)$. Si $p>1$, c'est l'espace des fonctions $f(x')$ telles que

$$\left(\int|f(x')|^p dx'\right)^{1/p} + \left(\iint|f(x')-f(y')|^p|x'-y'|^{-p-n+2}dx'dy'\right)^{1/p} < \infty.$$

J. L. Lions (Nancy)

1526:

Gagliardo, Emilio. Proprietà di alcune classi di funzioni in più variabili. Ricerche Mat. **7** (1958), 102–137.

Étude des espaces $W_p{}^r(\Omega)$ de Sobolev: fonctions u sur Ω ouvert de R^n, telles que $D^q u\in L_p(\Omega)$ pour $|q|\leq r$. L'A. retrouve de façon simple et élégante les principaux résultats de Sobolev, en les améliorant, notamment lorsque $p=1$. *J. L. Lions* (Nancy)

1527:

Da Silveira, Miguel. Correction to the article "General operational calculus in N variables". Portugal. Math. **16** (1957), 41–42.

Dans l'article "General operational calculus" [Portugal. Math. **15** (1956), 49–69; MR **18**, 747] l'A. avait accepté comme vrai que, si un espace localement convexe E est complet par rapport aux suites de Cauchy, il en est de même pour l'espace $\mathcal{L}(E)$ des applications linéaires continues de E dans lui-même, muni de la topologie de la convergence simple. Ce détail n'a aucune influence dans le calcul opérationnel des opérateurs D_i; mais, dans le cas général, on doit supposer $\mathcal{L}(E)$ complet.

En outre l'A. donne un énoncé plus précis de son théorème fondamental, dont l'énoncé antérieur prêtait à des confusions. *J. Sebastião e Silva* (Lisbon)

1528:

Singh, U. N. Topological vector space of entire functions. Math. Student **26** (1958), 83–91.

This paper is essentially a detailed resume of the results obtained by V. Ganapathy Iyer in a series of papers dealing with the structures of the linear space Γ of all entire functions, endowed with the natural topology τ of uniform convergence on compact sets. Rather than use τ directly, the author has chosen to follow Iyer in basing the treatment upon a specific metric for τ. Descriptions of the results discussed, and comments on these from the point of view of general linear space theory, are to be found in reviews of the relevant papers [Iyer, J. Indian Math. Soc. (N.S.) **12** (1948), 13–30; **17** (1953), 183–185; Quart. J. Math. Oxford Ser. (2) **1** (1950), 86–96; Proc. Amer. Math. Soc. **3** (1952), 874–883; **7** (1956), 644–649; MR **10**, 380; **15**, 719; **12**, 108; **14**, 657; **17**, 1225]. There is also a brief discussion of the notion of proper basis, as introduced in Γ by Arsove [Proc. Amer. Math. Soc. **8** (1957), 264–271; MR **19**, 259].

R. C. Buck (Los Angeles, Calif.)

1529:

Erohin, V. D. On conformal transformations of rings and the fundamental basis of the space of functions analytic in an elementary neighbourhood of an arbitrary continuum. Dokl. Akad. Nauk SSSR **120** (1958), 689–692. (Russian)

Unter einer elementaren Umgebung des Kontinuums K versteht Verfasser ein Gebiet $G\supset K$ mit der Eigenschaft, daß für jedes Berührgebiet D_q von G das Gebiet $G_q=G\cap D_q$ höchstens zweifach zusammenhängend ist. Er zeigt in Theorem 2, daß es zu K und G eine Doppelfolge in G regulärer Funktionen $e_{00}(z)\equiv 1$, $e_{qn}(z)$ $(n=1,2,\cdots)$ gibt, so daß jedes in G reguläre f auf genau eine Weise in der Form $f(z)=a_{00}+\sum_q\sum_{n=1}^\infty a_{qn}e_{qn}(z)$ (gleichmäßige und absolute Konvergenz in G) dargestellt werden kann. Dabei gelten verschiedene Abschätzungen, die unter anderem vom Modul R_q von G_q abhängen:

(1) $\sup_{z\in K}|e_{qn}(z)| < C(\delta)(1+\delta)^n$;

(2) $\sup_{z\in G} |e_{qn}(z)| < C(\delta)(1+\delta)^n R_q^n$;

(3) $|a_{qn}| < C(\delta)(1+\delta)^n R_q^{-n} \sup_{z\in G_q} |f(z)|$

(jeweils für beliebiges $\delta > 0$);

(4) $|a_{qn}| < C_q \sup_{z\in K} |f(z)|$

(falls K_q endliche Länge hat).—Diesen Satz hat der Verfasser in der Informationstheorie angewandt [#1530]. —In Theorem 1 wird die Funktion φ, die ein zweifach zusammenhängendes Gebiet auf einen Kreisring abbildet, zerlegt in der Form $\varphi(z)=\varphi^2(\varphi^1(z))$, wobei φ^1 und φ^2 schon in gewissen einfach zusammenhängenden Gebieten regulär sind.—Theorem 3 zeigt: Eine in K erklärte Funktion f ist genau dann in einem einfach zusammenhängenden Gebiet $G \supset K$ regulär, wenn sie auf K mit Hilfe gewisser Basisfunktionen $e_n(z)$ "genügend gut" approximiert werden kann. *K. Zeller* (Tübingen)

1530:

Erohin, V. D. Asymptotic theory of the ε-entropy of analytic functions. Dokl. Akad. Nauk SSSR **120** (1958), 949–952. (Russian)

Verfasser betrachtet den kompakten Raum $A_G{}^K(M)$ der Funktionen f, die in einem Gebiet G regulär und dem Betrage nach $\leq M$ sind, wobei die Norm von f durch das Betragsmaximum in einem Kontinuum $K \subset G$ erklärt ist. Mit $H_\varepsilon = H_\varepsilon(A_G{}^K(M))$ bezeichnet er die ε-Entropie dieses Bereiches, und zwar verwendet er unter den verschiedenen möglichen, im wesentlichen äquivalenten Definitionen folgende: $H_\varepsilon = \log N_\varepsilon = \log_2 N_\varepsilon$, wo N_ε die Minimalzahl der Punkte eines ε-Netzes ist [vergleiche Kolmogorov, dieselbe Dokl. **108** (1956), 385–388; MR **18**, 324; und Vituškin, ibid. **117** (1957), 745–747; MR **20** #3750]. Nach Kolmogorov ist $H_\varepsilon \asymp (\log M/\varepsilon)^2$; Verfasser fragt nun, wann hier sogar asymptotische Gleichheit besteht. Nach Theorem 1 gilt $H_\varepsilon(A_G{}^K(M)) \approx \tau(K/G)\,(\log M/\varepsilon)^2$ mit $\tau(K/G) = \sum_q 1/\log R_q$ unter folgenden Voraussetzungen: Es gibt unendlich viele Punkte, die nicht in G liegen; K enthält mehr als einen Punkt; jedes Komplementärgebiet D_q von K schneidet G in einem höchstens zweifach zusammenhängenden Gebiet D_q, dessen Modul R_q sei $(1 < R_q \leq +\infty$; ist G_q einfach zusammenhängend, so wird $R_q = +\infty$ gesetzt).—In Theorem 2 wird das auf Funktionen mehrerer Variablen übertragen, in Theorem 3 werden ganze Funktionen endlicher Ordnung betrachtet.—Der Beweis beruht auf einem früheren Ergebnis des Verfassers über Reihenentwicklungen analytischer Funktionen [#1529]. *K. Zeller* (Tübingen)

1531:

Vaĭnberg, M. M. A non-linear operator on Orlicz space. Studia Math. **17** (1958), 85–95. (Russian)

Let L^M be the Orlicz space corresponding to space L_M. Assume that function M satisfies a condition (termed a Δ_2-condition in this paper) so that $L_M = L^M$. [See A. Zygmund, *Trigonometrical series*, Dover, New York, 1955; MR **17**, 361; pp. 95–97, where L^M is denoted by $L_M{}^*$.] Let $g(u, x)$ be defined for all real u and for all $x \in B$ where B is a measurable subset of Euclidean s-space, and assume that g satisfies condition (H): g is continuous in u for almost all fixed $x \in B$; and for each fixed u, g is measurable on B. Define a mapping h from L_M into L_{M_1} in this way: if $u(x) \in L_M$, let $hu = g(u(x), x)$. Mapping h is said to be continuous at $u_0 \in L_M$ provided: if u_n converges to u_0 in the mean, i.e., if

$$\lim_{n\to\infty} \int_B M(|u_n(x)-u_0(x)|)dx = 0,$$

then hu_n converges to hu_0 in L_{M_1} in the mean. Theorem: Mapping h is continuous at each point of L_M if and only if

$$|g(u, x)| \leq a(x) + M_1^{-1}(bM(|u|)),$$

where $a(x) \in L_{M_1}$, M_1^{-1} is the inverse of M_1, and $b > 0$. *J. Cronin* (New York, N.Y.)

1532:

Subba Rao, M. V. Closure theorems. Math. Student **26** (1958), 61–70.

If S is a subset of a locally convex topological linear space E, and if the only continuous linear functional which vanishes on S is the zero functional, then S generates a dense subspace of E. In this expository paper, the author illustrates the use of this in proving several simple closure theorems for the spaces l^p, and certain related non-normed spaces of analytic functions. {Remark: The answer to the question which the author raises at the end of his paper is: no (e.g., $a_1 = 1$, $a_n = -1/2^{n-1}$). It should also be noted that since the space $\Gamma(R)$ is norm isomorphic to l^1, theorem 4 is subsumed by the stronger theorem 1.} *R. C. Buck* (Los Angeles, Calif.)

1533:

Feldzamen, A. N. A generalized Wehr characteristic. Bull. Amer. Math. Soc. **65** (1959), 79–83.

Two operators T_1 and T_2 defined on a Banach space X are similar if there exists a nonsingular operator L such that $T_2 = LT_1L^{-1}$. A complete set of similarity invariants is well known in two particular cases, i.e., if X is a Hilbert space and T_i $(i=1, 2)$ are normal (the multiplicity function) or if X is finite dimensional (the Wehr characteristics). The first step for a general theory of similarity is due to the present work. The author introduces in a natural way the Wehr characteristics for a spectral measure $E(\cdot)$ (of uniform multiplicity $n < \infty$) with regard to a quasi-nilpotent operator Q commuting with $E(\cdot)$. These Wehr characteristics are similarity invariants for a certain class (large enough) of spectral operators in Hilbert space. For such operators, the author's Wehr characteristics form a complete set of invariants for a more general equivalence (semi-similarity; a notion which seems to be the author's) than similarity. *C. Foiaş* (Bucharest)

1534:

Scroggs, James E. Invariant subspaces of a normal operator. Duke Math. J. **26** (1959), 95–111.

Let H be a Hilbert space and A a normal operator in H. A is said to have property (P) if each invariant subspace of A reduces A. The author's principal results are the following: 1° If the eigenvectors of A span H, then a necessary condition that A not have (P) is that there exists a set of eigenvalues $E = \{\lambda_n\}$ such that every point of E is inaccessible from the unbounded component of $\complement\bar{E}$. 2° If the spectrum of A contains a sequence of simple closed rectifiable Jordan curves $\{C_i\}$ such that C_i is in the bounded component of $\complement C_{i+1}$ $(i=1, 2, \cdots)$, then

(*P*) fails for *A*. Different applications to subnormal operators are also given. Methods of the theory of analytic functions are deeply used. *C. Foiaş* (Bucharest)

1535:

Butler, John. Perturbation series for eigenvalues of analytic non-symmetric operators. Arch. Math. **10** (1959), 21–27.

The author gives a method for calculating certain power series of projections which occur in the analytical perturbation of an operator in a Banach space.
C. Foiaş (Bucharest)

1536:

Ladyženskaya, O. A.; and Faddeev, L. D. On continuous spectrum perturbation theory. Dokl. Akad. Nauk SSSR **120** (1958), 1187–1190. (Russian)

The operator $L=L_0+K$ is considered on the space $H=\int_I{}^+ H_\lambda d\lambda$, where I is an interval of the real line, $H_\lambda=A$ for any λ, and $d\lambda$ is Lebesgue measure. L_0 is multiplication by λ and K is given by a kernel $k(\lambda,\mu)$ which must satisfy several continuity conditions as well as being a completely continuous self-adjoint operator on A for each $\lambda, \mu \in I$. Under suitable conditions, L is proved to be self-adjoint and to have a finite number of proper values of finite multiplicity (occurring either in I or exterior to it). If P is the orthogonal projection on the subspace spanned by these proper vectors, then $L(I-P)$ is unitarily equivalent to L_0. An application to quantum mechanics is discussed briefly.
R. R. Kemp (Kingston, Ont.)

1537:

Kuroda, Shige Toshi. On a theorem of Weyl-von Neumann. Proc. Japan Acad. **34** (1958), 11–15.

The author proves that if $\alpha(X)$ is a unitarily invariant cross norm, in the sense of Schatten, but not equivalent to the trace norm, then any self-adjoint transformation H on a separable Hilbert Space can be perturbed into an H' with pure point spectrum by the addition of an X of class α, i.e., $H'=H+X$. This is a generalization of the Weyl theorem in which α is the Schmidt norm. The work of T. Kato has shown that norms equivalent to the trace norm must be eliminated in the hypothesis.
F. J. Murray (New York, N.Y.)

1538:

Neveu, J. Théorie des semi-groupes de Markov. Univ. California Publ. Statist. **2** (1958), 319–394.

This work constitutes the elaboration of the author's thesis presented to the Université de Paris in 1955. It consists of three parts: I. Théorie des semi-groupes d'endomorphismes sur un espace de Banach; II. Théorie des semi-groupes de Markov; III. Semi-groupes de diffusion sur un intervalle.

Part I gives an exposition of semi-group theory refined in Bourbaki terminology. The author's essential contributions are expounded in §5, §7 and §9. §5: The convergence of a sequence of semi-groups is related to that of the resolvents of the corresponding infinitesimal generators. §7: A "generalized semi-group" of linear operators T_s^t ($T_s^u=T_s^tT_t^u$, $s\leq t\leq u$, $T_s^s=I$) on a Banach space X is related to a semi-group of linear operators T_a defined on the space of mappings f of $[0,\infty)$ into X by $T_af(s)=T_s^{s+a}f(s+a)$. §9: A linear operator A defined on a dense subspace of a Fréchet space X is the infinitesimal generator of a semi-group of linear operators T_t on X which is equi-continuous in t and strongly continuous at $t=0$, if and only if 1° the resolvent $(I-\lambda^{-1}A)^{-1}$ exists for $\lambda>0$ such that $(I-\lambda^{-1}A)^{-n}$ are equi-continuous for $\lambda>0$, $n\geq 0$, and 2° $(I-\lambda^{-1}A)^{-1}$ converges strongly to I as $\lambda\to\infty$.

Part II discusses in detail the two special problems: (i) the semi-group induced by a Markov process defined on a locally compact abelian group G such that the operators of the semi-group are commutative with the translations U_z ($(U_zf)(x)=f(x+z)$ where $x, z\in G$); and (ii) the semi-group induced by a Markov process in an n-dimensional euclidean space R^n. In the case (ii), a fairly general class of second-order elliptic integro-differential operators is derived as the infinitesimal generators of the semi-groups.

Part III: A general form of the infinitesimal generator of a semi-group of diffusion on an interval is derived by an approach which is entirely different from that by W. Feller [Ann. of Math. (2) **60** (1954), 417–436; MR **16**, 488]: Feller makes essential use of the "local character" of the infinitesimal generator, whereas the author starts with the Laplace transform of the "first passage probability" $P(x,y;u)[t]=\Pr\{\exists\theta\leq t;\ X(\theta)=y,\ X(s)\neq u$ when $s<\theta|X(0)=x\}$ of the path function $X(\theta)$; here $\exists$ denotes "there exists" and $(u-x)(x-y)>0$. *K. Yosida* (Tokyo)

1539:

Kurepa, Svetozar. On the continuity of semigroups of normal transformations in Hilbert space. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. **13** (1958), 81–87. (Serbo-Croatian summary)

The author proves the following two theorems, both of which establish strong continuity from very weak assumptions. Theorem 1: Let $[N(t);\ t\geq 0]$ be a semi-group of bounded normal operators defined on a Hilbert space and such that $N(t)f=0$ implies $f=0$ for each $t\geq 0$. Let T be a bounded closed set with $m(T+T)>0$. Then if $N(t)$ is weakly continuous on T it is necessarily strongly continuous for $t\geq 0$. Theorem 2: Let $[V(t);\ -\infty<t<\infty]$ be a group of bounded linear operators defined on a Hilbert space. If $V(t)$ is strongly continuous on a bounded perfect set T such that $m(T+T)>0$, then $V(t)$ is strongly continuous for all real t.
R. S. Phillips (Los Angeles, Calif.)

1540:

Elliott, Joanne. Absorbing barrier processes connected with the symmetric stable densities. Illinois J. Math. **3** (1959), 200–216.

The author studies the transition function of stable symmetric processes (having exponent less than 1) and the corresponding processes with absorbing barriers. She exhibits the infinitesimal generator of the related semi-groups on continuous or integrable functions and computes the expected time of absorption. For related work see Kac and Pollard [Canad. J. Math. **2** (1950), 375–384; MR **12**, 114] as well as Elliott and Feller [Trans. Amer. Math. Soc. **82** (1956), 392–420; MR **19**, 185]. {In footnote 3 the author mentions an alternative proof of relation (0.6) that requires the knowledge of a certain probability; this probability was found by H. P. McKean [Ann. of Math. (2) **61** (1955), 564–579; MR **16**, 1036; result (7.1)].} *G. A. Hunt* (Ithaca, N.Y.)

1541:

Ehrenpreis, L.; and Mautner, F. I. Some properties of the Fourier-transform on semisimple Lie Groups. III. Trans. Amer. Math. Soc. **90** (1959), 431–484.

This paper continues the discussion of Fourier analysis on the group G of all conformal mappings of the unit disc begun in Parts I and II [Ann. of Math. (2) **61** (1955), 406–439; Trans. Amer. Math. Soc. **84** (1957), 1–55; MR **16**, 1017; **18**, 745]. The authors determine the maximal closed two-sided ideals of three convolution algebras of distributions on G: the algebra D of all infinitely differentiable functions with compact support, the algebra E' of all distributions with compact support, and the algebra $L^1(G)$. For each complex number s, one may define a representation $U(\cdot, s)$ of G on L^2 of the unit circle by the formula $(U(g, s)a)(z) = |dgz/dz|^s a(gz)$; when $\Re s = \frac{1}{2}$, then $U(g, s)$ is unitary. If f is a distribution on G, it is natural to refer to $\mathfrak{F}(s) = \int U(g, s)f(g)dg$ as the Fourier transform of f, at least for those complex numbers s for which the integral makes some reasonable sense. If f is in D or E', its Fourier transform makes sense for all complex s; if f is in $L^1(G)$, its Fourier transform makes sense in the strip $0 \leq \Re s \leq 1$. Each non-integral complex number s determines a maximal closed ideal in D and in E' consisting of all the elements whose Fourier transforms vanish at s. If s is an integer, then $U(\cdot, s)$ is a sort of blend of three representations: two representations $V^{\pm}(\cdot, s)$ from the discrete series of irreducible unitary representations, and a finite dimensional irreducible representation. Correspondingly, each integer determines three maximal closed ideals in D and E'. Any closed ideal in D or E' is determined by the values of s where the Fourier transforms vanish together with the order of vanishing (with appropriate threefold complication at integers s). Each s in the strip $0 \leq \Re s \leq 1$ with $s \neq 0$ and $s \neq 1$ determines a maximal closed ideal in $L^1(G)$ consisting of the functions whose Fourier transforms vanish at s. Each integer l determines two maximal closed ideals in $L^1(G)$ consisting of those functions whose representating operators relative to $V^{\pm}(\cdot, l)$ vanish. Finally, there is the maximal closed ideal in $L^1(G)$ of all functions whose integrals are 0. An example is given of a closed ideal in $L^1(G)$ contained in no maximal closed ideal. The authors also define a one-parameter family of convolution algebras of functions whose Fourier transforms they can characterize, and obtain some results about their ideal structure. All these results are applied to obtain further information about mean periodic and two-sided mean periodic functions as defined in Part II.

Let Γ be any discrete subgroup of G such that G/Γ is compact. It is shown that the regular representation of G on $L^2(G/\Gamma)$ decomposes discretely, the multiplicity of an irreducible unitary representation U of G being determined as follows (cf. Frobenius reciprocity). Let H be the representation space of U, and let $\{e_n\}$ be a basis of eigenvectors for a maximal compact subgroup (isomorphic to the circle) of G. Let H^0 be the linear submanifold of vectors with rapidly decreasing coordinates, topologized in the appropriate topology for rapidly decreasing sequences. Then the multiplicity of U is the dimension of the subspace of fixed vectors under Γ in the dual space to H^0. The authors also prove an analogue of the Poisson summation formula for Γ.

In the final section a one-to-one correspondence is established between a certain class of modular forms and a certain class of pairs, the first component of which is a meromorphic function H satisfying the functional equation

$$\pi^{-s}\Gamma(s)H(s) = \pi^{s-1}\Gamma(1-s)H(1-s)$$

and the second component of which is a rapidly decreasing sequence. *W. F. Stinespring* (Princeton, N.J.)

1542:

Hartman, S. Beitrag zur Theorie des Massringes mit Faltung. Studia Math. **18** (1959), 67–79.

The author reproves the following result. If $a(n) = \int_{-\pi}^{\pi} e^{inx}d\alpha(x)$ $(n = 0, \pm 1, \pm 2, \cdots)$ where $d\alpha(x)$ is a finite regular measure on the Borel subsets of $(-\pi, \pi]$, and if $d\alpha$ is without continuous singular part, then $\inf |a(n)| > 0$ implies that $a(n)^{-1} = \int_{-\pi}^{\pi} e^{inx}d\beta(x)$ where $\beta(x)$ is a finite regular measure, etc. This result was first proved by Beurling [9e Congrès des Math. Scand. Helsingfors, 1938, pp. 345–366, 1939], using a rather different method. It has also been proved by Gel'fand, Raikov, and Šilov [Uspehi Mat. Nauk. (N.S.) **1** (1946), no. 2 (12), 48–146; MR **10**, 258], whose method is substantially identical with that employed in the present paper.

I. I. Hirschman, Jr. (St. Louis, Mo.)

1543:

Curtis, Philip C., Jr. Order and commutativity in Banach algebras. Proc. Amer. Math. Soc. **9** (1958), 643–646.

Let A be a real banach algebra with identity e such that $\|e\| = 1$. Let C be the closure of finite sums of squares of elements of A. Theorem: If C lattice orders A, then A is commutative. The same result holds for A a banach *-algebra with a continuous involution and an identity, and C the closure of finite sums of elements xx^*. This constitutes an extension of a theorem by the reviewer [Amer. J. Math. **73** (1951), 227–252; MR **13**, 47].

S. Sherman (Philadelphia, Pa.)

1544:

Pârvu, Monica Pavel. Sur des espaces linéaires. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **7** (1958), no. 20, 27–31. (Romanian. Russian and French summaries)

The resolvent equation is discussed in a quasi-normed algebra, that is, the norm may be p-homogeneous for some p with $0 < p \leq 1$. *M. M. Day* (Urbana, Ill.)

1545:

Segal, I. E. Distributions in Hilbert space and canonical systems of operators. Trans. Amer. Math. Soc. **88** (1958), 12–41.

The author discusses absolute continuity of distributions on a Hilbert space. He applies the results to the question of when pseudo-canonical transformations of Bose-Einstein or Fermi-Dirac canonical systems are genuine canonical transformations. Also he classifies all Bose-Einstein canonical systems over a Hilbert space.

A distribution on a real Hilbert space is a linear map of a dense submanifold into not necessarily commuting random variables. An abelian distribution on a finite dimensional space amounts simply to a probability measure on it. Theorem 2 gives an effective criterion for the absolute continuity of one infinite product distribution with respect to another in terms of the convergence of a numerical infinite product. {There is a misprint in this

product; its factors ought to be $E_{n_\lambda}[(dm_\lambda/dn_\lambda)^{(1/2)}]$.} This result tells just when an affine transformation of the normal distribution yields an absolutely continuous one.

Theorem 5 describes the most general Bose-Einstein canonical system over a real Hilbert space $\mathfrak{H}$. If m is a quasi-invariant abelian distribution on $\mathfrak{H}$, one may form a Bose-Einstein canonical system from two unitary representations of $\mathfrak{H}$ on $L_2(\mathfrak{H}, m)$: the multiplication representation and the regular representation. Two such systems are unitarily equivalent if and only if the distributions are mutually absolutely continuous. Every Bose-Einstein system is a direct sum of ones related to the foregoing type.

Corollaries 5.1 and 5.2 discuss what transformations of a Bose-Einstein system of the foregoing type are genuinely canonical (i.e., implementable by a unitary operator) when the distribution maps an orthonormal basis into independent random variables. Corollary 5.3 is the analogue of corollary 5.2 for the most important Fermi-Dirac canonical system. {In the proofs, "Theorem 5" appears several times where "Theorem 4" is meant.} These corollaries give quite specific information in the case of the free field systems, as is shown by examples.

W. F. Stinespring (Princeton, N.J.)

1546:

Feldman, Jacob. Equivalence and perpendicularity of Gaussian processes. Pacific J. Math. **8** (1958), 699–708.

Let L be a linear space of real-valued functions, μ and ν two probability measures on the σ-field determined by L. If all the functions of L are Gaussian random variables for both μ and ν then it is shown that either μ and ν are equivalent (each absolutely continuous with respect to the other) or they are perpendicular (mutually singular). Let K be the linear span of L and the real constants. Then μ and ν are equivalent if and only if the μ-equivalence classes of K are the same as the ν-equivalence classes of K and the identity correspondence between the $L_2(\mu)$-closure of K and the $L_2(\nu)$-closure of K is a bounded invertible operator T such that $\sqrt{(T^*T)}-I$ is a Hilbert-Schmidt operator. This criterion is due to I. E. Segal (see below). Both the equivalence-perpendicularity dichotomy and the equivalence criterion generalize a case of S. Kakutani's theorem on equivalence of infinite product measures [Ann. of Math. (2) **49** (1948), 212–224; MR **9**, 340]. The proof essentially reduces the problem to a question of product measures (corresponding to a reduction of $\sqrt{(T^*T)}-I$ to principal axes). {The mimeographed notes of I. E. Segal referred to in the paper have now been published [#1545 above]. The fact referred to before Lemma 6 is proved on p. 15 of Segal's paper. Lemma 1 in the present paper is incorrectly stated, but this is easily remedied by replacing "self-adjoint" by "positive" and, in part (b), $(A-I)^2$ by A^2-I.}

E. Nelson (Princeton, N.J.)

1547:

Stinespring, W. Forrest. Integration theorems for gages and duality for unimodular groups. Trans. Amer. Math. Soc. **90** (1959), 15–56.

This paper may be recognized as a continuation of the Segal's paper [Ann. of Math. (2) **57** (1953), 401–457; MR **14**, 991] whose notations and terminologies are used in this paper. Let $\Gamma=(\mathfrak{G}, \mathscr{A}, m)$ be a gage space in the sense of Segal. The author introduces two kinds of convergences for the gage space Γ such that: a sequence of measurable operators $\{A_n\}$ converges in measure [resp. grossly] to a measureable A, if for given $\varepsilon>0$ [resp. for given $\varepsilon>0$ and given $T\in L_1(\Gamma)$] there exists a sequence $\{P_n\}$ of projections in $\mathscr{A}$ such that (1) $\|(A_n-A)P_n\|<\varepsilon$ $(n=1, 2, \cdots)$ and (2) $m(I-P_n)\to 0$ [resp. (1) and (2′) for a given $\varepsilon_1>0$ there exists an integer $N>0$ such that each $I-P_n$ $(n>N)$ satisfies $|m(TQ)|<\varepsilon_1$ for every projection $Q\lesssim I-P_n$]. The convergence in measure is a formal extension of that of usual measure theory. Then the usual relations to convergence nearly everywhere (n.e.) and mean convergence are given and several dominated convergence theorems are proved together with an extension of Fatou's lemma. Furthermore the author proves that their convergences preserve addition, adjoint * and multiplication, where for the convergence-in-measure case the proof for multiplication is done under a bounding condition. As he states, convergence either n.e. (in the sense of Segal) or in measure implies the gross one, and when Γ is a probability gage space, i.e., the case $m(I)=1$, the convergences in measure and grossly are equivalent. For such Γ, several convergence-in-measure theorems in probability measure space are extended. The author introduces quadratic functional, by which is meant a function q from the Hilbert space H to $[0, +\infty]$ (including $+\infty$) satisfying Hilbert space norm condition. Using this q, he generalizes the monotone convergence theorem of Segal. Applying the quadratic functional, he extends a result of Segal's with respect to Fubini theorem and discusses the tensor product of two gage spaces. The tensor product is extended to that of closed, densely defined operators on two Hilbert spaces.

In the last part, the author develops the harmonic analysis for a unimodular locally compact group $\mathfrak{G}$. Denote by L_a $(a\in\mathfrak{G})$ and L_f $(f\in L_1(\mathfrak{G}))$ the left regular representations of $\mathfrak{G}$ and of $L_1(\mathfrak{G})$ on $L_2(\mathfrak{G})$, and denote by $\mathscr{L}$ the von Neumann algebra generated by them and by m the gage of $\mathscr{L}$ introduced by the Haar measure of $\mathfrak{G}$. Let W be a unitary operator on $L_2(\mathfrak{G}\times\mathfrak{G})$ $(=L_2(\mathfrak{G})\otimes L_2(\mathfrak{G}))$ defined by $(Wf)(x, y)=f(x, xy)$ and $\Phi(\cdot)$ the operation $T\to W^{-1}(T\otimes I)W$ for each $T\in\mathscr{L}$. The author defines the convolution multiplication $*$ in $L_1(\Gamma)$ $(\Gamma=(L_2(\mathfrak{G}), \mathscr{L}, m))$ by $m((F*G)T)=(m\times m)((F\otimes G)\Phi(T))$ for each $T\in\mathscr{L}$; then it becomes a commutative Banach algebra with respect to the convolution multiplication, $L_1(\Gamma)$-norm, and with involution $F\to\bar{F}$ which is defined by usual conjugation; and further if $\mathfrak{G}$ is abelian then it is just the L_1-group algebra of the character group of $\mathfrak{G}$. The back (or namely inverse) transform of $F\in L_1(\Gamma)$ is defined by $f(a)=m(L_{a^{-1}}F)$ and it transforms the convolution * in $L_1(\Gamma)$ to pointwise multiplication in $\mathfrak{G}$. Further, he discusses the relation between operators $F\in L_1(\Gamma)$ and the functions $f(a)$ of $\mathfrak{G}$ under the back transform, and the Riemann-Lebesgue lemma is proved. In the final part, the author shows a duality of $\mathfrak{G}$, which coincides with Pontrjagin duality when $\mathfrak{G}$ is abelian. The theorem is summarized as the following: if Γ' denotes the set of $U\in\mathscr{L}$ satisfying $U\otimes U=\Phi(U)$ and $U=\bar{U}$, then there is a one-to-one correspondence between Γ' and the set ($\mathscr{C}$ say) of *-homomorphisms of $L_1(\Gamma)$ into the complex numbers, the correspondence being given by $S\in\Gamma'\leftrightarrow\sigma_S(\cdot)$ $(=m(S^*\cdot))\in\mathscr{C}$, and further Γ' is the unitary group consisting of just the operators $\{L_a; a\in\mathfrak{G}\}$, and the mapping $a\to L_a$ is homeomorphically group-isomorphism from $\mathfrak{G}$ onto Γ', where the topology in Γ' is the weak operator one.

H. Umegaki (Tokyo)

1548:

Tomiyama, Jun. Generalized dimension function for W^*-algebras of infinite type. Tôhoku Math. J. (2) **10** (1958), 121–129.

A multiplicity theory similar to that of Pallu de la Barrière [Bull. Sci. Math. France, **82** (1954), 1–51; MR **16**, 491], is given for rings of operators of infinite type. A coupling invariant is introduced and used to give conditions for spatial isometry. Similar results are also given by Kadison [Ann. of Math. (2) **54** (1951), 326–338; MR **13**, 256] and Griffin [Trans. Amer. Math. Soc. **75** (1953), 471–504; **79** (1955), 386–400; MR **15**, 539; **17**, 66].

E. L. Griffin, Jr. (Ann Arbor, Mich.)

1549:

Saitô, Teishirô. On incomplete infinite direct product of W^*-algebras. Tôhoku Math. J. (2) **10** (1958), 165–171.

The author extends J. von Neumann's discussion [Compositio Math. **6** (1938), 1–79] of incomplete infinite direct products to those whose algebraic factors are not necessarily full operator algebras. Cases in which type III algebras as well as non-normal algebras occur are discussed.

E. L. Griffin, Jr. (Ann Arbor, Mich.)

1550:

Turumaru, Takasi. Crossed product of operator algebra. Tôhoku Math. J. (2) **10** (1958), 355–365.

The operator algebras constructed by Murray and von Neumann [Ann. of Math. (2) **37** (1936), 116–229] are shown to be equivalent to certain crossed product algebras defined by the author. In particular, the author proves: "Let A be a unitary algebra and G a group of $*$-automorphisms which preserve the inner-product invariantly, then the crossed-product (A, G) is also a unitary algebra."

E. L. Griffin, Jr. (Ann Arbor, Mich.)

1551:

Šragin, I. V. Weak continuity of the Nemyckiĭ operator. Moskov. Oblast. Pedagog. Inst. Uč. Zap. **57** (1957), 73–79. (Russian)

M. M. Vaĭnberg [Dokl. Akad. Nauk SSSR **73** (1950), 253–255; **92** (1953), 213–216; MR **12**, 111; **15**, 439] studied norm continuity from L^p to $L^{p'}$ of what he called Nemyckiĭ's operator h:

$$hu(x) = g(u(x), x),$$

where $g(u, x)$ is a function of two variables continuous in $-\infty < u < \infty$ for almost every x in some measurable set B in E_n, and measurable in B for each u. This note shows that h is weakly continuous from $L^p(B)$ to $L^{p'}(B)$ if and only if $g(u, x) = a(x) + b(x)u$, with suitable conditions on a and b.

M. M. Day (Urbana, Ill.)

1552:

Vaĭnberg, M. M.; and Èngel'son, Ya. L. A conditional extreme of functionals in linear topological spaces. Mat. Sb. N. S. **45 (87)** (1958), 417–422. (Russian)

Let φ be a functional with non-zero differential at a point x_0 in a linear topological space with a total set of linear functionals, and let U be the level surface $\varphi^{-1}(\varphi(x_0))$. This note states and proves a sufficient condition that if f is a functional differentiable at x_0 which takes at x_0 an extreme of its values on U, then the gradient of f at x_0 is a scalar multiple of the gradient of φ at x_0.

M. M. Day (Urbana, Ill.)

CALCULUS OF VARIATIONS

See 1445, 1733.

GEOMETRY

See also 1609.

1553:

Deaux, R. Sur la focale de van Rees. Mathesis **67** (1958), 209–213.

Let $q = abcd$ denote a complete quadrilateral and $a_1 = (ad, bc)$, $b_1 = (bd, ca)$, $c_1 = (cd, ab)$ the sides of its diagonal trilateral $a_1b_1c_1$.

The locus of the foci of the conics of the tangential pencil determined by q is a cubic Γ passing through 13 "remarkable" points of q.

The author associates with q the six complete quadrilaterals aba_1b_1, cda_1b_1; bcb_1c_1, adb_1c_1; cac_1a_1, bdc_1a_1. By considering the pencils of conics determined by these quadrilaterals he obtains twenty-four additional points of Γ whose geometric determination is simple.

The author is also led to the consideration of the Moebius involution and obtains, among others, the following result.

If two conics of a tangential pencil have a common axis of symmetry containing the two couples of real foci and if those two couples separate each other, all the couples of real foci of the conics of the pencil lie on that axis and belong to an involution; the imaginary foci lie on an ideal circle having for diameter the segment determined by the double points E, F of the involution—the centers of the (imaginary) circles of the pencil.

If the two initial couples of foci do not separate each other, E, F are real, and the conics having their centers on the segment EF have their real foci on the circle having EF for diameter.

The circles of the pencil may be either real or ideal; in both cases, the couples of foci belong to the Moebius involution having E, F for double points.

N. A. Court (Norman, Okla.)

1554:

Hofmann, Jos. E. Zur elementaren Dreiecksgeometrie in der komplexen Ebene. Enseignement Math. (2) **4** (1958), 178–211.

Feuerbach's circle, Euler's and Wallace's lines, Holz's and Morley's theorems and other properties of the triangle are here obtained elegantly by choosing the circumcircle as the unit circle in the complex plane.

F. A. Behrend (Melbourne)

1555:

Dubikajtis, L. La géométrie de Lie. Rozprawy Mat. **15** (1958), 112 pp.

By this memoir (his dissertation for the degree of candidate) the author intends to fill a gap in the literature, giving an axiomatization and the beginnings of a synthetic

treatment of Lie's geometry of spheres in n dimensions. Principal reference is to Blaschke's *Vorlesungen über Differentialgeometrie*, vol. III, chap. 5–6 [Springer, Berlin, 1929], where an essentially analytic exposition of Lie's geometry in 2 and 3 dimensions can be found. The space of Lie's geometry is the n-dimensional space M_n of Möbius, i.e. the euclidean E_n completed by a single point at infinity which is supposed to lie on every straight line, but on no proper circle. The best known model $\mathfrak{M}_n$ of Lie's geometry is the Möbius space M_n in which (1) the system $\mathscr{L}$ of all oriented K-spheres (cf. Blaschke, loc. cit., p. 226) and (2) the notion of contact of two K-spheres a and b, here denoted by $a-b$, have been introduced. The geometry proper is then defined, according to Klein's Erlanger Programm, by the corresponding group of transformations, in the present case the "group of Lie", consisting of all invertible mappings of $\mathfrak{M}_n$ onto itself, carrying K-spheres into K-spheres and preserving contact.

For a convenient reproduction of the axioms, some of the author's symbols and notations have to be explained. Contact relations are of the greatest importance throughout the work. The relation of two systems $(a_1, \cdots, a_k)$, $(b_1, \cdots, b_l)$ in contact, i.e., $a_i - b_j$ for all $i=1, \cdots, k$, $j=1, \cdots, l$, is briefly expressed by $(a_1, \cdots, a_k)-(b_1, \cdots, b_l)$. The symbol $(a_1, \cdots, a_k)_{\not-}$ means that $a_i \not- a_j$ for all $i \neq j$. Generalizing Blaschke's "Vorzeicheninvariante": $\delta(a_1, \cdots, a_k) = -1$ if there is at least one sphere touching all $a_1, \cdots, a_k$, $=+1$ if there is no such sphere, which in the case $k=3$ and $(a_1, a_2, a_3)_{\not-}$ is the only invariant of three K-spheres. If $(a_1, a_2, a_3)_{\not-}$ is not satisfied then all contacts between a_1, a_2, a_3 form a complete system of invariants of the three K-spheres. The $a_1, \cdots, a_k$ are dependent on $(b_1, \cdots, b_l)$ if, for each c such that $c-(b_1, \cdots, b_l)$, also $c-(a_1, \cdots, a_k)$; the K-spheres $a_1, \cdots, a_k$ are said to be dependent if and only if at least one of them depends on the others. A system of K-spheres $(a_1, \cdots, a_k)$ represents a simplex α_k if $k \geqq 2$, if the $a_1, \cdots, a_k$ are independent, and if two b_1, b_2 exist such that $b_1 \not- b_2$ and $(b_1, b_2)-(a_1, \cdots, a_k)$. A simplex α_2 of two elements is called a couple. Two couples $\alpha_2 = (a_1, a_2)$, $\beta_2 = (b_1, b_2)$ are said to be connected if they satisfy the conditions $\delta(a_1, a_2, b_1, b_2) = -1$ and each of the a_1, a_2, b_1, b_2 depends on the others; in symbols $\alpha_2 \bigcirc \beta_2$. If $\alpha_2 \bigcirc \beta_2$ and $a_1 - b_1$ or $a_1 - b_2$ then one writes $\alpha_2 \ominus \beta_2$; if, however, $\alpha_2 \bigcirc \beta_2$ and $(a_1, a_2, b_1, b_2)_{\not-}$ then $\alpha_2 \oplus \beta_2$. Two simplexes α_k, β_l separate each other, in symbols $\alpha_k \mid \beta_l$, if for every a and b the conditions $a - \beta_l$, $b - \alpha_k$ imply $a = b$.

Chap. 1 is of a preliminary nature. Here the axioms are stated as theorems on K-spheres and contact in $\mathscr{L}$ and proved. Thus their acceptance as axioms for an abstract synthetic development of Lie's geometry is suggested and, assuming $\mathfrak{M}_n$ as a reality, their consistency is established.

Chap. 2: 'Axioms and general theorems of Lie's geometry'. It is noted that there are few mathematical theories with a smaller number of primitive notions: (1) the system $\mathscr{L}$ of the K-spheres, subsequently called "elements"; (2) contact, both not defined at this stage, but in their interrelation described by the axioms. These are now formulated again (with "element" instead of "K-sphere" and in slightly changed wording); they are divided in five groups:

I. Axioms of contact: (1) Each element a satisfies $a-a$. (2) $a-b$ implies $b-a$. (3) There are two a, b such that $a \not- b$.

II. Axioms on triplets: (1) $\delta(a, b, c) = -1$ if and only if either a, b, c form a simplex or $(a, b, c)_{\not-}$ is not satisfied. (2) If a, b, c are different and dependent, then $a-b$, $b-c$, $a-c$. (3) If (a, b, c), (a, b, d), (a, c, d) are simplexes, then $\delta(b, c, d) = -1$.

III. Axioms of dimension (depending on n): (1) The conditions (i) $k+l=n+3$, (ii) $\alpha_k - \beta_l$, (iii) $\alpha_k - b_0$, (iv) $a_0 - \beta_l$ imply the contact $a_0 - b_0$. (2) If $k+l<n+3$ and $a_1, \cdots, a_k$ form a simplex touching β_l, then there is an element a_{k+1} touching β_l and such that the elements $a_1, \cdots, a_k, a_{k+1}$ are independent. (3) The system $(a_1, \cdots, a_n, a_{n+1})$ is a simplex if and only if (i) $a_1, \cdots, a_n$ form a simplex and (ii) there are two elements b_1, b_2 touching $(a_1, \cdots, a_n)$ and such that $\delta(b_1, b_2, a_{n+1}) = +1$.

IV. Axiom of connection (liaison): If for three couples $\alpha_2, \beta_2, \gamma_2$ one has (i) $\alpha_2 \bigcirc \beta_2$, (ii) $\beta_2 \bigcirc \gamma_2$, and (iii) there is an element d touching α_2 and γ_2, but not β_2; then $\alpha_2 \bigcirc \gamma_2$.

V. Axiom of separation (or continuity): Let $\mathscr{A}$, $\mathscr{B}$ be two subsets of $\mathscr{L}$ and assume that (i) there are elements $a_0, b_0 \in \mathscr{L}$ such that for each $a \in \mathscr{A}$ and each $b \in \mathscr{B}$ one has $(a_0, b) | (b_0, a)$; (ii) there are $a_1 \in \mathscr{A}$, $b_1 \in \mathscr{B}$ such that a_0, a_1, b_1 form a simplex; then there is an element c_0 such that, for each $a \in \mathscr{A}$ and for each $b \in \mathscr{B}$ different from c_0, one has $(b_0, c_0) | (a, b)$.

The rest of Chap. 2 is taken up by a large number of theorems, some of which are of simple nature and not explicitly proved, more or less immediate consequences of the axioms, as, e.g., Theorem 2.12: There are three elements a, b, c such that $\delta(a, b, c) = +1$ (implying that there is no element touching a and b and c). One of the more complicated statements is, e.g., Theorem 2.26: Let $k \geqq 1$ and (i) $\beta_2 - (a_1, \cdots, a_k, a_{k+1})$; (ii) $a_1, \cdots, a_k$ independent; (iii) $a_1, \cdots, a_k, a_{k+1}$ dependent; then a_{k+1} depends on $(a_1, \cdots, a_k)$. (Proof by induction.)

In Chap. 3, 'System of reference. Points, hyperplanes and hyperspheres', use is made of Theorem 2.12: Let $(N, \bar{k}, \underline{k})$ be of the kind that no element touches $N, \bar{k}, \underline{k}$. Then N is called "the point at infinity", and the couple $\kappa_2 = (\bar{k}, \underline{k})$ the "fundamental hypersphere", $\bar{k}$ its outer, $\underline{k}$ its inner side. An element a is called a point if there is a simplex α_2 such that $(a, N) - \alpha_2 \oplus \kappa_2$. A couple α_2 is said to be conjugate if there is a point simplex β_{n+1} touching α_2; the two elements of α_2 are called its sides. The conjugate couple α_2 is called a hyperplane if it touches N, a hypersphere if it does not touch N. Theorems are proved to justify these definitions. The following chapters serve to demonstrate that the notions introduced by these definitions are actually isomorphic to those in $\mathfrak{M}_n$, built up from the space of metric geometry. Chap. 4, 'Manifolds. Incidence axioms of affine geometry', begins with the definition of a k-dimensional manifold: the empty set if $k=-2$, a single point if $k=-1$, the set of all elements depending on a simplex α_{k+2} if $0 \leqq k \leqq n-1$, and the set of all points if $k=n$. A k-dimensional manifold is denoted by a^k and its dimension $d(a^k) = k$. Defining the union $a^k \cup b^l$ of two manifolds a^k and b^l as the smallest manifold containing a^k and b^l, the intersection $a^k \cap b^l$ as the greatest manifold contained in a^k and b^l, it is shown that the system of all manifolds represents a lattice with respect to $\cup$ and $\cap$, and the partial order relation $\subset$. Moreover the incidence axioms of affine geometry are established.

Chap. 5, 'Axioms of order, of parallelism, and of continuity of affine geometry', defines contact of manifolds a^k, b^l: Either $a^k \subset b^l$, or, if $k \leqq l$: $d(a^k \cap b^l) = -1$ and $d(a^k \cup b^l) = l+1$. In the case that a^k and b^l are planes and in contact, they are said to be parallel: $a^k \| b^l$. It is proved that through every finite point A there is a unique a^1

parallel to a given line b^1. A point B is said to lie between A and C (in symbols: $A|B|C$) if and only if A, B, C are finite points and $(A, C)|(B, N)$. Finally Hilbert's order axioms and Dedekind's continuity axiom are established in the geometry of the manifolds.

Chap. 6, 'Metric', shows that (1) "hyperspheres", i.e., manifolds a^k not containing N, are quadrics, and therefore ellipsoids; (2) all hyperspheres are similar. By distinguishing one hypersphere, one turns the affine space into a metric space model where the elements are the usual points, hyperplanes, hyperspheres. In Chap. 7 the categoricalness of the system of axioms is proved; thus there is essentially one Lie geometry, all possible models are isomorphic. Finally the author states that he is preparing papers dealing with the independence of the axioms and with the further development of Lie's geometry.

H. Schwerdtfeger (Montreal, P.Q.)

1556:

Tănăsescu, Aurelian. La méthode des paratraces dans le système de projection Monge. Bul. Inst. Politehn. Bucureşti **19** (1957), no. 3/4, 69–77. (Romanian. Russian and French summaries)

Elementary constructions (plane sections of prisms, pyramids, cylinders, cones) in classical descriptive geometry.

O. Bottema (Delft)

1557:

Šindelář, Karel. Conic sections. Pokroky Mat. Fys. Astr. **4** (1959), 145–156. (Czech)

Entirely elementary discussion of the quadratic equation in two non-homogeneous real or complex coordinates with obvious geometrical interpretation.

H. Schwerdtfeger (Montreal, P.Q.)

1558:

Myller, A. Triangles remarquables du point de vue centro-affine. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **9** (1958), no. 1, 1–4. (Romanian. Russian and French summaries)

"Interprétation des triangles d'aire maximale inscrits dans une ellipse, comme triangles équilatéraux dans la géométrie centro-affine. Interprétation en sens inverse des triangles qui seraient rectangulaires dans cette géométrie."

Résumé de l'auteur

1559:

Leisenring, Kenneth. A theorem on nonloxodromic Möbius transformations. Michigan Math. J. **6** (1959), 51–52.

If a Möbius transformation T has a fixed circle g, it has a coaxal family of fixed circles. It is shown that if T maps the points A, B, C of g on A', B', C', then the axis of the family is the Pascal line of the hexagon $AB'CA'BC'$.

F. A. Behrend (Melbourne)

1560:

Martynenko, V. S. Some metric geometries the absolute of which is a curve of the third order. Ukrain. Mat. Ž. **10** (1958), 251–269. (Russian. English summary)

The author derives the line element for some metric geometries with respect to a cubic curve: Neyl's parabola, the cissoid, Newton's trident, Descartes' leaf, the curves $y^3-x^3+3x^2=0$ and $y(y^2-x)=0$. All computations are made in inhomogeneous coordinates and concern eight geometries which are, in fact, defined by two connexes, for which—presumably without being aware of this—the author has chosen two of the three comitants of $(a'x)^3$ or mixed polars of them:

$$\theta = (a'b'u')^2(a'x)(b'x), \quad \varphi = (a'b'u')^2(a'c'u')(b'x)(c'x)^2,$$
$$F = (a'b'u')^2(a'c'u')(b'd'u')(c'd'u')^2.$$

Nearly all of the author's computations, made with great perseverence, are superfluous if one uses the appropriate Clebsch-Weitzenböck symbols, which makes it, moreover, immediately clear that the author has obtained only partial results in various cases. As an example we indicate that for the metric based on θ, the x-discriminant of θ, i.e. $(Fu')^6=0$, provides the minimal "distance zero" curves, which are the tangents to the cubic f, whereas the u'-discriminant gives the "angle zero" minimal curves as

$$((a'b')(c'd')x)^2(a'x)(b'x)(c'x)(d'x) = \tfrac{4}{3}H\cdot f = 0.$$

This last curve consists, contrary to the author's results, of the curve $f=0$ and its Hessian $H=(a'b'c')^2(a'x)(b'x)(c'x)$. The Hessian is lacking in the paper. Correspondingly, the results obtained for the connex Q are incomplete. Here, because of the Cayley identity for binary cubic forms, the minimal curves depend essentially on θ, and $f=0$ is the envelope of the minimal distance curves and the Hessian the locus of their nodes.

As the complete system of the ternary cubic form consists of 31 non-trivial comitants, over 400 geometries can be defined using a cubic curve as fundamental curve. The author did not mention this and gave no reason for choosing only the eight special combinations, but it seems to the reviewer that using concomitants of higher degree would lead to unfeasable computations along the lines the author has chosen.

E. M. Bruins (Amsterdam)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 1299.

1561:

★**Hadwiger, H. Vorlesungen über Inhalt, Oberfläche und Isoperimetrie.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. xiii + 312 pp. DM 46.20.

This book, based on the author's lectures at the University of Bern, is essentially a survey of measure theory and its implications for convex bodies in k-dimensional Euclidean space E_k. The author's aim is to keep his discussion as elementary as possible, and with this restriction, the book is remarkably complete. Prerequisites are a thorough acquaintance with k-dimensional affine geometry, a knowledge of analysis as contained in a contemporary advanced calculus course, and some familiarity with the algebra of sets. The book is written in somewhat the same spirit as, but contains much more material than, Mayrhofer's *Inhalt und Mass* [Springer-Verlag, Wien, 1952; MR **14**, 733]. It is totally different in its outlook from Halmos's *Measure theory* [Van Nostrand, New York, 1950; MR **11**, 504]. It may remind the reader of I. M. Yaglom and V. G. Boltyanskiĭ's *Convex figures* [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951; MR **14**, 197], although it is much more advanced and covers much more ground.

Chapter 1. Elementary geometry of polyhedra. This chapter is introductory. Definition and simplest properties of polyhedra in E_k; special types of polyhedra; vector addition of polyhedra; invariance under various groups of motions; invariance under dissections. (Throughout the book, invariance of classes of sets and of various functionals under different groups of motions is heavily emphasized.) Chapter 2. Elementary content. The existence and uniqueness of a translation-invariant, additive, nonnegative content Φ on the class of polyhedra such that Φ is 1 for the unit cube are proved. A source of possible confusion is the author's use of the same notation for both simplices and cubes (pp. 4 and 35). The reviewer would have been happy with more details, in the proof of the existence theorem, Satz II, page 39, that the functional Φ is well-defined. This chapter is noteworthy for the purely elementary arguments employed and for the explicit formula given for the content of an arbitrary polyhedron. Chapter 3. Jordan content and Lebesgue measure. Jordan content for bounded sets is obtained from the content for polyhedra defined previously, and its properties are exhaustively studied. The reasons for this remain obscure to the reviewer. Lebesgue measure is obtained from Jordan content by a definition differing slightly from the usual one. The computations employed in obtaining the usual properties of Lebesgue measure are long and somewhat tedious. Various facets of measure theory, e.g. the Banach-Tarski paradox and extensions of Lebesgue measure, are discussed without proofs. Chapter 4. Selected studies in set-geometry. Jung's inequality and Gale's inequality; metrization of the space of compact subsets of E_k; Brunn's theorem; various symmetrizations of compact subsets of E_k. Chapter 5. Content, surface area, and isoperimetry. Minkowski's relative surface area; the Brunn-Minkowski inequality; the isoperimetric theorem as an inequality for relative surface areas and measures of two arbitrary compact subsets of E_k; isoperimetry of the sphere. Chapter 6. Convex bodies and general integral geometry. This chapter contains a large number of results on functionals for convex bodies, Blaschke's integral geometry, and so on. The definition on page 225 of the integral seems to the reviewer to be incomplete.

The book is copiously provided with informative notes, placed at the ends of the chapters. There is also an extensive bibliography. The reader's convenience has been considered in apparently every conceivable way, even to the delicate touch of providing back references from the notes to the main text. This book will provide a valuable reference work for experts in the field as well as a highly readable, if demanding, introduction for would-be experts.

E. Hewitt (Seattle, Wash.)

1562:

Lee, Shen-Ling. Some properties of a regular oval. Advancement in Math. **2** (1956), 153–160. (Chinese. English summary)

In this paper, we give the relation of the curvature and the angle formed by the tangent and the chord of an arc of a regular oval. From this relation we may easily get the following properties of a regular oval.

1. The necessary and sufficient condition of the arc of a regular oval is that there exists a circle touching the arc at two distinct points.

2. Let $A_1, A_2, \cdots, A_{2n}$ be the vertices of a regular oval where $A_1, A_3, \cdots, A_{2n-1}$ have the minimum of curvatures and $A_2, A_4 \cdots, A_{2n}$ the maximum of curvatures. Then the angles of the inscribed polygon $A_1A_2 \cdots A_{2n}$ have the following property:

$$\angle A_1 + \angle A_3 + \cdots + \angle A_{2n-1} > \angle A_2 + \angle A_4 + \cdots + \angle A_{2n}.$$

3. Let $A_1, A_2, \cdots, A_{2n}$ be as above. The tangents at the vertices form a $2n$-sided polygon $B_1B_2 \cdots B_{2n}$ where B_i is the point of intersection of the tangents at A_i and A_{i+1}, $i=1, 2, \cdots, 2n$, $A_{2n+1}=A_1$, and B_i and the arc $\widehat{A_iA}_{i+1}$ are on the same side of the line A_iA_{i+1}. Then

$$B_{2n}B_1 + B_2B_3 + \cdots + B_{2n-2}B_{2n-1} > B_1B_2 + B_3B_4 + \cdots B_{2n-1}B_{2n}.$$

4. The vertices of an oval (non-circle) are not all on the same circle.

From the author's summary

1563:

Lee, Da-tsin. On ovals of n-type. Advancement in Math. **3** (1957), 623–627. (Chinese. English summary)

Let $\rho(\varphi)$ be the radius of curvature of an oval C, $a_1, a_2, \cdots, a_k, \cdots$ and $b_1, b_2, \cdots, b_k, \cdots$ be the Fourier coefficients of $\rho(\varphi)$. C is said of n-type if $a_1=a_2=\cdots=a_n=b_1=b_2=\cdots=b_n=0$ and a_{n+1}, b_{n+1} are not both zero. In the present paper we mainly obtain the following theorem: An oval of n-type has at least $\left(\frac{n+1}{2}\right)+\Delta$ squares circumscribed to it, where

$$\left(\frac{n+1}{2}\right) = \begin{cases} \dfrac{n+1}{2} & \text{if } n \text{ is odd,} \\[2ex] \dfrac{n+2}{2} & \text{if } n \text{ is even;} \end{cases}$$

$$\Delta = \begin{cases} 0 & \text{if } \left(\dfrac{n+1}{2}\right) \text{ is odd,} \\[2ex] 1 & \text{if } \left(\dfrac{n+1}{2}\right) \text{ is even.} \end{cases}$$

Author's summary

1564:

Laugwitz, Detlef. Eine Bemerkung über konvexe Kurven. Math. Z. **70** (1958/59), 463–464.

If K is a closed, convex, bounded, plane region it is shown that if P is an interior point of K that at least two parallelograms exist with P as a vertex two sides of which are within support lines of K. This is an alternative proof of a result due to Stein [Math. Z. **68** (1957), 282–283; MR **19**, 877]. The present proof is almost an immediate consequence of a representation of the support lines of K given by Radon.

D. Derry (Vancouver, B.C.)

1565:

Stein, Sherman K. A continuous mapping defined by a convex curve (addendum). Math. Z. **70** (1958/59), 465.

If K is a closed, bounded, convex region in a plane it is shown that four points P, Q, R, S, taken in order, exist on the boundary of K so that the lines through P and R parallel to QS as well as those through Q and S parallel to PR all support K. This leads to an alternative proof of the

author's result [Math. Z. **68** (1957), 282–283; MR **19**, 877] that there is at least one point T interior to K so that at least four parallelograms exist with T as a vertex, two sides of which are contained in support lines of K.

D. Derry (Vancouver, B.C.)

1566:

Grünbaum, B. On common transversals. Arch. Math. **9** (1958), 465–469.

A family P of subsets of the plane is said to have the property T(n) [property T] if any n members [all members] of P have a common transversal. Then: (1) If P is a family of disjoint translates of a parallelogram, then T(5) implies T; here T(5) can not be replaced by T(4); (2) if P is a family of congruent circles containing at least six members, then T(4) implies T. Conjecture: Theorem (1) probably holds for families of disjoint translates of any convex set.

L. A. Santaló (Buenos Aires)

1567:

Pleijel, A. Über Distanzpotenzintegrale. Arch. Math. **9** (1958), 430–432.

Let K be a convex body in n-dimensional Euclidean space, P and Q be variable points in K, dV_P and dV_Q the corresponding volume-elements, and r be the distance PQ. The author considers the integral $D=\int r^m dV_P dV_Q$, where $m>-n$, and attempts to prove that among all convex bodies with the same hyperplane-measure the sphere has the greatest integral D. However, his argument involves a lemma that is incorrect in the cases required. {For example, let

$$F(\vartheta)=\left(\int_{-\pi}^{\pi}\{1+(\vartheta-\tfrac{1}{2})\sin\phi\}^2 d\phi\right)^{-p}.$$

Then, according to the lemma, $F''(\vartheta)$ would be non-negative on [0, 1] for sufficiently large positive values of p, whereas in fact $F''(\frac{1}{2})=-3p(2\pi)^{-p}<0$.}

H. P. Mulholland (Exeter)

GENERAL TOPOLOGY, POINT SET THEORY

See also 1277, 1367, 1522, 1603.

1568:

Young, G. S. A generalization of Bagemihl's theorem of ambiguous points. Michigan Math. J. **5** (1958), 223–227.

Let D be the interior of the unit $(n-1)$-sphere S^{n-1} in E^n and f a function from D into a compact metric space M. A closed r-cell I in the boundary of D is called an r-cell of disjoint cluster sets for f if I is the intersection of the boundary of D with the boundary of either of two $(r+1)$-cells J_1 and J_2 in D such that there do not exist sequences of points $\{p_k\}$ in J_1-I and $\{q_k\}$ in J_2-I for which $\lim p_k=\lim q_k$ and $\lim f(p_k)=\lim f(q_k)$. Theorem: S^{n-1} does not contain uncountably many disjoint $(n-2)$-cells of disjoint cluster sets of f.

R. H. Fox (Princeton, N.J.)

1569:

Kirkor, A. Wild 0-dimensional sets and the fundamental group. Fund. Math. **45** (1958), 228–236.

A compact 0-dimensional subset A of S^3 is called tamely imbedded if it is contained in some tame arc of S^3, otherwise wildly imbedded. The well-known Antoine set is obviously wildly imbedded because its complement is not simply connected. The author gives an example of a compact 0-dimensional subset A of S^3 that is wildly imbedded even though its complement is simply connected.

R. H. Fox (Princeton, N.J.)

1570:

Császár, Ákos. Sur la notion d'espace topologique. I, II, III. Mat. Lapok 8 (1957), 37–60, 211–231; 9 (1958), 37–63. (Hungarian. Russian and French summaries)

This is a well written introduction to general topology. The author's treatment is well organized, and easy to read. A new feature of the presentation is that topological, proximity, and uniform spaces are considered in close connection. The contents of these three papers are the following.

Metric spaces. Topological spaces. T_0, T_1, T_2-spaces. Proximity spaces. Uniform spaces. Pseudo-metric spaces. Generalized Urysohn lemma. Compact spaces. Products of spaces. The Tychonoff product theorem. Embedding theorems for T_0-spaces and for completely regular spaces. Smirnoff's embedding theorem on proximity spaces; it admits a generalization by a method of S. Mrówka. Introduction of proximity structures in completely regular spaces, uniform structures in proximity spaces, pseudometric in uniform spaces. A. H. Stone's theorem on the paracompactness of metric spaces. The Urysohn metrization theorem and the Nagata-Smirnoff metrization theorem. Metrization of uniform spaces.

J. Erdős (Debrecen)

1571:

Hiyashi, Eiichi. A topology defined by limit points. Sûgaku **9** (1957/58), 149. (Japanese)

The author considers a way of defining a closure of a set in a separable regular space by joining its condensation points and proves that the space turns out to be a Urysohn space.

H. Yamabe (Osaka)

1572:

Mrówka, S. On the unions of Q-spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 365–368.

In Bull. Acad. Polon. Sci. Cl. III **5** (1957), 947–950 [MR **20** #1967], the author showed that if X is a normal space that is the union of countably many closed subspaces that are Q-spaces in the sense of Hewitt [Trans. Amer. Math. Soc. **64** (1948), 45–99; MR **10**, 126], then X is a Q-space, and showed by example that the hypothesis that X be normal cannot be deleted. In the present article, the author gives an example of a completely regular (non-normal) non-Q-space that is the union of two closed subspaces that are Q-spaces.

M. Henriksen (Lafayette, Ind.)

1573:

Mrówka, S. An example of a non-normal completely regular space. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 161–163.

An example is given of a completely regular non-normal space X that is a countable union of closed para-compact subspaces. Indeed, X is a Q-space (cf. the review above), and has a countable dense subset. [Examples of this sort

are known; cf. E. Michael, Proc. Amer. Math. Soc. **4** (1953), 831–838; MR **15**, 144.]

M. Henriksen (Lafayette, Ind.)

1574:

Mamuzić, Zlatko. Sur la solution d'un problème concernant eT-espaces. Glasnik Mat.-Fiz. Astr. Ser. II. **11** (1956), 95–103. (Serbo-Croatian summary)

Soit T un tableau ramifié d'ensembles; désignons par eT le V-espace correspondant défini par les voisinages $V(x)$ que voici: si $\gamma x \in' \mathrm{II}$, c'est-à-dire si l'ordinal γx de l'ensemble $(\cdot, x)$ des prédécesseurs de $x \in T$ n'est pas un ordinal-limite, alors $V(x)=\{x\}$; si $\gamma x \in \mathrm{II}$, alors $V(x)$ est de la forme $(z, \cdot) \setminus (x, \cdot)$ avec $z<x$ [cf. Kurepa, Thèse, Paris, 1935; p. 67]. L'espace eT est définissable par l'écart abstrait que voici; $f(a, a)=\gamma a$ ou $\gamma a+2$ suivant que $a \in \mathrm{II}$ ou $a \in' \mathrm{II}$; $f(x, y)_{\neq}=t(\cdot, x] \cap (\cdot, y]$ [cf. #1575 ci-après]. Il s'agit alors [ibid., prob. 4.1.1] de caractériser la condition $O^{1\prime}$ que voici: $a \in \bar{X} \to f(a, a) \in \overline{f(X, X)}$ $((a), X \subseteq T$; $f(F, F)=\{f(x, x) | x \in X\})$. Un eT-espace définissable par l'écart précédent satisfait à la condition $O^{1\prime}$ de continuité si et seulement si pour chaque $x \in T$, $\gamma x \in \mathrm{II}$ il existe un $z_x<x$ vérifiant $(y \in R_{\gamma x}T \wedge y>z_x) \to y=x$ (Th. 7). Cette condition équivaut à ce que le dérivé de $R_\alpha T=\{x | x \in T, \gamma x=\alpha\}$ est vide pour chaque ordinal α respectivement que $R_\alpha T$ est disjoint du dérivé de $\bigcup R_\xi T$ $(\xi \geqq \alpha)$ (Th. 5).

Đ. Kurepa (Zagreb)

1575:

Kurepa, Đuro. Sur l'écart abstrait. Glasnik Mat.-Fiz. Astr. Ser. II. **11** (1956), 105–134. (Serbo-Croatian summary)

Étant donné un espace topologique généralisé M, l'A. appelle M-espace tout espace topologique généralisé E tel qu'il existe une fonction $\rho(a, b)$, appelée M-écart ou M-proximité, définie sur $E \times E$, dont les valeurs $\in M$, et vérifiant, pour tous éléments $a \in E$ et $b \in E$ et pour tout ensemble $F \subset E$, les axiomes O^1, O^2 et O^3 suivants:

$$O^1: \rho(a, b) = \rho(a, a) \Rightarrow a = b;$$
$$O^2: \rho(a, b) = \rho(b, a);$$
$$O^3: a \in \bar{F} \Leftrightarrow \rho(a, a) \in \overline{\rho(a, F)};$$

avec: $\rho(a, F)=$ ensemble des $\rho(a, x)$ pour tous les éléments $x \in F$.

L'A. n'impose pas sa condition de continuité [imposée par lui antérieurement, C. R. Acad. Sci. Paris **203** (1936), 1049–1052], à savoir: $O^{1\prime}: a \in \bar{F} \Rightarrow \rho(a, a) \in \overline{\rho(F, F)}$, avec: $\rho(F, F)=$ ensemble des $\rho(x, x)$ pour tous les éléments $x \in F$.

L'A. étudie spécialement le cas où M est un espace totalement ordonné ayant un premier (ou dernier) élément ζ et où on a: $\rho(a, a)=\zeta$ pour chaque élément $a \in E$, la topologie généralisée de M étant définie à partir de l'ordre de M [cf. Kurepa, Thèse, Paris, 1935; p. 11]. Il appelle M-écart totalement ordonné un tel M-écart.

L'A. démontre alors que de nombreux espaces considérés par lui et par d'autres auteurs dans des travaux antérieurs, peuvent être considérés comme des M-espaces, et souvent comme des M-espaces à M-écart totalement ordonné et même bien ordonné.

Il établit des cas de totale ordonnabilité de certains de ces espaces.

Il termine en posant divers problèmes concernant les espaces envisagés. *A. Appert* (Angers)

1576:

Shirai, Tameharu. A remark on the ranked space. II. Proc. Japan Acad. **33** (1957), 139–142.

[Pour la première partie, voir même Proc. **32** (1956), 120–124; MR **17**, 1065.]

L'A. démontre le théorème de Baire dans un espace rangé complet ne satisfaisant pas à l'axiome (C) de Hausdorff (mais vérifiant l'axiome (B)) en admettant une définition modifiée de la notion d'ensemble non-dense.

A. Appert (Angers)

1577:

Okano, Hatsuo. Some operations on the ranked spaces. I. Proc. Japan Acad. **33** (1957), 172–176.

L'espace produit d'espaces rangés est complet si et seulement si chaque espace coordonnée est complet.

A. Appert (Angers)

1578:

Okano, Hatsuo. On closed subspaces of the complete ranked spaces. Proc. Japan Acad. **33** (1957), 336–337.

Complétion des sous-espaces d'un espace rangé complet.

A. Appert (Angers)

1579:

Okano, Hatsuo. On the completion of the ranked spaces. Proc. Japan Acad. **33** (1957), 338–340.

Étude et résultats nouveaux concernant le problème de la complétion d'un espace rangé. *A. Appert* (Angers)

1580:

Slye, John Marshall. Collections whose sums are two-manifolds. Duke Math. J. **24** (1957), 275–298.

Let S be a compact 2-dimensional manifold and let f be a continuous mapping of S onto a simple arc Y. If each $f^{-1}(y)$ is a simple arc then S is a 2-cell; if each $f^{-1}(y)$ is a simple closed curve then S is either an annulus, a Möbius strip or a Klein bottle. *R. H. Fox* (Princeton, N.J.)

1581:

Keldyš, Lyudmila. Zero-dimensional open mappings. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 165–184. (Russian)

The author gives a detailed proof of her factorization theorem for light open dimension-raising mappings of finite-dimensional compact metric spaces [Dokl. Akad. Nauk SSSR **98** (1954), 719–722; MR **16**, 502] and construction of an example [ibid. **97** (1954), 201–204; MR **16**, 60]. She shows without finite-dimensionality that a light dimension-raising mapping taking closures of open sets to closures of open sets has the form gh, where g preserves the dimension of every open set and h is one-to-one except for a set whose image is first category (in other words, h is irreducible).

J. Isbell (Lafayette, Ind.)

1582:

Ponomarev, V. Multivalued mappings of topological spaces. Dokl. Akad. Nauk SSSR **124** (1959), 268–271. (Russian)

Let X, Y be topological spaces (satisfying appropriate separation axioms). Multivalued mappings f of X onto Y are considered (for any $x \in X$, fx is a subset of Y; every

$y \in Y$ lies in some fx); it is supposed that every fx is closed in Y, and every $f'y = \{x | y \in fx\}$ is closed in X. A mapping f is continuous [cf. V. Ponomarev, same Dokl. **118** (1958), 1081–1084; MR **20** #6685] if and only if $f'N$ is closed for any closed set $N \subset Y$; if $f'N$ is open for any open $N \subset Y$, then f is called skew continuous; if both conditions hold, then f is said to be strongly continuous. Closed [open] mappings are defined in the usual way. If every fx [every $f'y$] is compact, then f is called Y-compact [X-compact].

Some typical results may be restated as follows. If f is closed continuous Y-compact, then there exists a closed continuous mapping g of ωX onto $\omega Y (\omega X, \omega Y$ are Wallman compactifications) such that $gx = fx$ for $x \in X$. If X, Y are normal, f is strongly continuous Y-compact, then f has an extension which is a strongly continuous mapping of βX onto βY (this assertion is given as a corollary to a theorem concerning multivalued mappings of proximity spaces). If f is a "perfect", i.e. closed continuous, X- and Y-compact mapping, then X is locally compact if and only if Y is so. If f is an open perfect mapping and X is paracompact, then Y is paracompact.

There are various other results concerning mappings which preserve different properties of spaces, maps of "π-compact" (i.e., possessing a base consisting of sets with compact boundaries) spaces, etc. No proofs are given.

M. Katětov (Prague)

ALGEBRAIC TOPOLOGY

See also 1312, 1402.

1583:

★Godement, Roger. Topologie algébrique et théorie des faisceaux. Actualités Sci. Ind. no. 1252. Publ. Math. Univ. Strasbourg. no. 13. Hermann, Paris, 1958. viii+283 pp.

Cet excellent livre est présenté par l'auteur comme la première partie d'un ouvrage en deux volumes, qui pourrait prendre les dimensions d'un véritable traité de Topologie algébrique (considérée d'un point de vue différent du point de vue "géométrique" d'Eilenberg-Steenrod). Le but essentiel de ce premier volume est de traiter de la cohomologie d'un espace topologique quelconque à coefficients dans un faisceau, à l'aide des notions-clés de "faisceau flasque" et "faisceau mou" (qui se substituent avantageusement aux "faisceaux fins", cf. ci-dessous), et aussi de préparer les matières du second volume (opérations de Steenrod, etc...), par l'étude détaillée de la théorie des produits et des structures simpliciales.

Le premier des deux chapitres que comporte le livre traite, en suivant Cartan-Eilenberg, de l'algèbre homologique dans les catégories de modules sur un anneau unitaire. L'auteur donne dans le premier paragraphe de ce chapitre I toutes les définitions concernant les catégories, les foncteurs, les catégories additives et abéliennes, mais préfère renvoyer à D. Buchsbaum et A. Grothendieck pour l'exposé général de l'algèbre homologique dans les catégories abéliennes. Les foncteurs Ext en théorie des faisceaux sont traités à part dans le dernier paragraphe du chapitre II. Le paragraphe 3 du chapitre I est un riche et vigoureux exposé des complexes simpliciaux au sens d'Eilenberg-Zilber. Une large place y est faite aux produits: cartésien, tensoriel, cup-produit. On peut seulement regretter que les structures semi-simpliciales, renvoyées à un bref appendice à la fin du volume, n'aient pas été incorporées à ce paragraphe.

Le second chapitre est consacré à la théorie des faisceaux et occupe les deux-tiers du livre. Un faisceau $\mathscr{F}$ sur un espace X est dit flasque si toute section de $\mathscr{F}$ au-dessus d'un ouvert de X peut se prolonger à X tout entier. Tout faisceau $\mathscr{F}$ peut se plonger dans un faisceau flasque (par exemple, le faisceau $\mathscr{C}^0(X; \mathscr{F})$ des germes de sections non nécessairement continues de $\mathscr{F}$). Dans les espaces paracompacts, il est important d'avoir aussi la notion plus faible de faisceau mou: $\mathscr{F}$ est mou si toute section de $\mathscr{F}$ au-dessus d'un fermé se prolonge à l'espace tout entier. $\mathscr{F}$ est fin si le faisceau Hom $(\mathscr{F}; \mathscr{F})$ est mou.

A chaque faisceau $\mathscr{A}$ correspond une "résolution flasque" canonique, foncteur exact de $\mathscr{A}$:

$$0 \to \mathscr{A} \to \mathscr{C}^0(X; \mathscr{A}) \to \mathscr{C}^1(X; \mathscr{A}) \to \cdots$$

où

$$\mathscr{C}^n(X; \mathscr{A}) = \mathscr{C}^0(X; \mathscr{Z}^n(X, \mathscr{A})),$$
$$\mathscr{Z}^n(X, \mathscr{A}) = \mathscr{C}^{n-1}(X, \mathscr{A}) / \mathscr{Z}^{n-1}(X, \mathscr{A}).$$

On pose alors par définition:

$$H^n(X; \mathscr{A}) = H^n[\Gamma(\mathscr{C}^*(X; \mathscr{A}))].$$

Les théorèmes fondamentaux [cf. Séminaire Cartan] s'en déduisent à l'aide de suites spectrales. A l'aide de ceux-ci il est ensuite prouvé, a posteriori, que l'on peut définir la cohomologie de X à coefficients dans A par une résolution flasque quelconque de $\mathscr{A}$.

Citons des questions importantes traitées en détail: suite exacte de cohomologie associée à un sous-espace fermé; dimension cohomologique; relations entre les groupes $H^n(X, \mathscr{A})$ et $\check{H}^n(X, \mathscr{A})$ obtenus par la méthode de Čech (bien entendu reliés par une suite spectrale), et cas d'isomorphismes, entre les groupes $H^n(X, \mathscr{A})$ et $H^n(U, \mathscr{A})$, où U est un recouvrement, et théorème de Leray.

Le § 6 étend à la théorie des faisceaux les notions de produit cartésien et de cup-produit. Il y est construit pour tout faisceau $\mathscr{A}$ une autre résolution flasque, foncteur exact de $\mathscr{A}$:

$$0 \to \mathscr{A} \to \mathscr{F}^0 \to \mathscr{F}^1 \to \cdots$$

avec

$$\mathscr{F}^n(X; \mathscr{A}) = \mathscr{C}^0(\mathscr{F}^{n-1}(X; \mathscr{A}))$$

dont la différentielle (donnée par une formula du type d'Alexander) résulte d'une structure semi-simpliciale, ce qui prépare la définition des opérations de Steenrod en théorie des faisceaux que l'auteur traitera dans le second volume.

Le livre, sans doute d'un accès assez ardu pour les lecteurs qui ne sont pas prêts à se mesurer avec une suite spectrale, est étoffé de nombreux exemples, empruntés à la topologie algébrique, à la géométrie algébrique, aux variétés analytiques et différentiables. L'ampleur des idées, la clarté et l'efficacité de l'exposé en font un excellent ouvrage.

A signaler l'absence d'un index terminologique, qui, bien que la table des matières remplisse partiellement cet usage, aurait sans doute été apprécié des lecteurs.

R. Deheuvels (Bourg-la-Reine)

1584:

Gann, Dan-yen. Cohomology of locally compact and paracompact space. Sci. Record (N.S.) **2** (1958), 11–14.

An author who sets up a theory of sheaf cohomology normally proves a uniqueness theorem. [See, e.g., #1583;

p. 183.] One such theorem is due to I. Fáry [C. R. Acad. Sci. Paris **237** (1953), 552–554; MR **15**, 147]. It assumes that the underlying space is locally compact and that the supports are compact. The present paper shows that Fáry's proof can be extended to the case in which the space is paracompact and the supports are closed.

J. F. Adams (Cambridge, England)

1585:

★**Vesentini, Edoardo. Construction géométrique des classes de Chern de quelques variétés de Grassmann complexes.** Colloque de topologie algébrique, Louvain, 1956, pp. 97–120. Georges Thone, Liège; Masson & Cie, Paris, 1957. 375 fr. belges; 3000 fr. francais.

The Chern classes of the (complex) Grassmann varieties can be determined either as a very special case of the Chern classes of certain complex homogeneous spaces [see review below] or by extension of the argument used in § 13 of F. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie* [Springer-Verlag, Berlin, 1956; MR 18, 509] to determine the Chern classes of the bundle along the fibres of a $\mathbf{GL}(q, \mathbf{C})$-bundle with fibre $\mathbf{P}_{q-1}$. In this paper a third method is indicated and carried out for some special cases. The idea is as follows. Consider the canonical embedding $f:\mathbf{G}_{n,d}\rightarrow\mathbf{G}_{n+1,d}$, where $\mathbf{G}_{n,d}$ denotes the Grassmann variety of the $\mathbf{C}_d$'s in $\mathbf{C}_n$. The corresponding normal bundle ξ is the dual bundle of the universal sub-bundle η over $\mathbf{G}_{n,d}$, hence its Chern classes are, up to signs, elementary Schubert classes. Now if for example the Chern classes of $\mathbf{G}_{n+1,d}$ are known, those of $\mathbf{G}_{n,d}$ can be found by the Whitney duality formula, since the homology structure of $\mathbf{G}_{n,d}$ is known in terms of elementary Schubert classes and since f^* is known too. Conversely, since in certain dimensions f^* is an isomorphism, knowledge of the Chern classes of $\mathbf{G}_{n,d}$ gives certain Chern classes of $\mathbf{G}_{n+1,d}$. In this way $c_1(\mathbf{G}_{n,d})$ and $c_2(\mathbf{G}_{n,d})$ are generally determined, using an induction argument that starts with the known Chern classes of $\mathbf{P}_n$ and using furthermore some geometrical properties of Grassmann varieties. In the same way the total Chern classes $c(\mathbf{G}_{2,5})$, $c(\mathbf{G}_{2,6})$ and $c(\mathbf{G}_{3,7})$ are calculated.

By the methods of Borel and Hirzebruch a very simple general formula is obtained, but it would require some formal work to give complete explicit expressions in elementary Schubert classes. The author's results do not provide any general formula, but give in some particular cases a simple expression for the Chern classes of $\mathbf{G}_{n,d}$ directly in terms of Schubert classes.

The reviewer does not understand the proof of theorem V, which says that the bundles ξ and η are equivalent, but a proof of this fact can be obtained easily by using the method of *Neue topologische Methoden*, referred to above. Furthermore, the author does not worry about signs of Chern classes, but also these can be made precise without any essential change of method.

A. van de Ven (Leiden)

1586:

Borel, A.; and Hirzebruch, F. Characteristic classes and homogeneous spaces. I. Amer. J. Math. **80** (1958), 458–538.

This paper brings the beginning of a thorough exposition of the techniques in the cohomology theory of fiber bundles developed during the past years by the two authors. The primary impetus of the paper stems from the concrete problem of determining the characteristic classes of homogeneous manifolds. However the methods, apart from solving this problem in certain cases, have considerably wider applicability. This arises from the fact that the authors investigate their question in a more general setting, which can be roughly summarized as follows: Let $E\rightarrow B$ be a principal differentiable bundle, with structural Lie group G. If then U is a closed subgroup of G, we can consider the associated bundle $G/U\rightarrow B$ with fiber G/U. Now relations between the characteristic classes of B, E/U and the fiber G/U, when specialized to the case B a point, lead to explicit formulae for G/U.

The first purpose of the paper is to establish the relation between the characteristic classes of G/U and the weights of the isotropy representation of U on the tangent space of the identity coset of U.

The following special example will have to suffice as an indication of the type of results which are obtained. Let G be a compact Lie group with maximal torus $T\subset G$. Let W be the group of automorphisms of T induced by inner automorphisms of G—the Weyl-group of G on T. Let A be the symmetric algebra built on $H^1(T; Q)$, i.e., the polynomial algebra $Q[x_1, \cdots, x_s]$ where the x_i are a base for $H^1(T; Q)$. Clearly W acts on A. Let $\mathcal{J}$ be the ideal generated by the invariants of W in A of dimension >0. Then according to A. Borel and Leray, $H^*(G/T; Q)$ is isomorphic to $A/\mathcal{J}$ as a graded ring, if the degrees in A are multiplied by 2. Now let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{t}$ the Lie algebra of T. The isotropy representation of T on $\mathfrak{g}/\mathfrak{t}$ splits this space into invariant 2-planes, $\mathfrak{g}/\mathfrak{t}=e_1+e_2+\cdots+e_m$, on which T is represented nontrivially by rotations. Let w be an orientation of the planes e_i, $i=1, \cdots, m$. As was known before and is again shown here, every such orientation induces an almost complex structure on the tangent bundle of G/T, which in this structure will be denoted by $\mathcal{T}_w$. The Chern classes of $\mathcal{T}_w$ are then well defined and the question arises how they are determined in the ring $A/\mathcal{J}$.

Now the orientation w defines a homomorphism of T onto S^1, in a fixed orientation, for each $i=1, \cdots, m$. (If U_i denotes the kernel of the representation of T on e_i, the orientation w can be interpreted as an isomorphism of T/U_i with S^1.) Hence w determines m characters (the roots of G) $\alpha_i\in\operatorname{Hom}(T; S^1)$. In view of the natural isomorphism $\operatorname{Hom}(T; S^1)\approx H^1(T; Z)$ these roots can be interpreted as elements of A. The answer to our earlier problem is then that the total Chern class of $\mathcal{T}_w$ is given by $\prod_{i=1}^{m}(1+\alpha_i)$.

The following summary of the various pertinent chapters is essentially taken from the authors' own introduction. This reviewer, at least, was unable to improve on it.

Chapter I. The first three sections give a survey of standard properties of roots and linear representations; § 4 gives two characterizations of systems of positive roots which will be used in chapter IV. In § 5 we introduce the roots of a Lie group with respect to a commutative subgroup of type $(2, 2, \cdots, 2)$, which will occur in the description of Stiefel-Whitney classes.

Chapter II recalls those concepts of fibre bundle theory which are most often used in this paper, such as restriction and extension of the structural group (with respect to homomorphisms), integration over the fibre, and the bundle of vectors tangent to the fibres of a bundle whose typical fibre admits a differentiable structure invariant under the structural group, to be called hereafter the "bundle along the fibres".

Chapter III starts with a review of the definition by means of symmetric functions of the characteristic classes of a bundle having a classical group as structural group (§ 9). In § 10, the λ-extension η^1 of a principal G-bundle η by means of a unitary representation $\lambda: G \to U(n)$ is considered; the bundle η^1 is a principal $U(n)$-bundle, whose Chern classes are shown to be the elementary symmetric functions in the weights of λ, suitably interpreted as 2-dimensional classes; analogous statements are proved for the real orthogonal representations and Pontrjagin classes (10.3). Now, the real tangent bundle to G/U is the i-extension of the principal U-bundle $(G, G/U, U)$, with respect to the linear isotropy representation i of U in the tangent space at a point of G/U fixed under U (proposition 7.5). Applying 10.3 to this situation yields the relation between roots and characteristic classes mentioned at the beginning of this introduction, which, in fact, holds more generally for the bundle along the fibres of $(E/U, B, G/U)$, where (E, B, G) is a principal G-bundle.

Chapter IV. The homogeneous space G/U admits an invariant almost complex structure J if and only if the isotropy representation i can be factorized through the standard inclusion of $U(n)$ in $SO(2n)$, $(2n = \dim G/U)$; one obtains in this case a unitary representation $i_c: U \to U(n)$, whose weights are certain roots of G, to be called the roots of J; they allow one to compute the Chern classes of J using 10.3 and to discuss the integrability of J using the results of § 4; among the applications new proofs of some results of H. C. Wang are given in § 13.

The invariant complex structures of G/U (where U is the centralizer of a torus in G), can be obtained directly by using the complexification of G; the space G/U is also homogeneous kählerian and even rational algebraic, and there is a close connection between its projective embeddings and the linear representations of G. For later use, a short discussion of some of these results is included in § 14; moreover, it is proved (14.10) that the real cohomology classes of these algebraic homogeneous spaces are all of type (p, p), and for $p=1$ those which are positive in the sense of Kodaira are described.

Chapter V is devoted to some special cases; in particular to projective spaces. *R. Bott* (Cambridge, Mass.)

1587:

★**Bott, R.; and Samelson, H. Applications of Morse theory to symmetric spaces.** Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 282–284. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv + 334 pp.

This is the sketch of the results of the authors concerning the homology of the loop space Ω of a Riemannian manifold M on which a compact, connected Lie group K acts in a variationally complete manner. The detailed account has appeared, under the same title, in Amer. J. Math. **80** (1958), 964–1029. Let p be the point based on which Ω is defined, and N an orbit of K. To each transversal geodesic segment s from p to N, a manifold $K(s)$ and a mapping $f_s: K(s) \to \Omega$ are defined. Let $\bar{k}(s)$ denote the f_s-image of the fundamental cycle mod 2 of $K(s)$, and if $K(s)$ is orientable, let $k(s)$ denote the f_s-image of the fundamental integral cycle of $K(s)$. The following are proved: (I) If p lies on an orbit of the highest dimension and K is variationally complete, then the set of $\bar{k}(s)$ forms a base of $H_*(\Omega; Z_2)$. If, moreover, all $K(s)$ are orientable, then the set of $k(s)$ forms a base of $H_*(\Omega, Z)$. (II) If $M = G/K$ is a symmetric Riemannian space, then both the action of K on M and the action of K on the tangent space of M at the fixed point are variationally complete.

Using the above results, the authors calculate the Poincaré series mod 2 of the loop space of symmetric spaces. In the case that the symmetric space is a group space, the integral cohomology ring $H^*(K(s))$ of $K(s)$ is explicitly given. *H. C. Wang* (Evanston, Ill.)

1588:

Bott, Raoul. The stable homotopy of the classical groups. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 933–935.

The author announces and gives a sketch of the following theorem:

(1) $\pi_k(U) = \pi_{k+2}(U)$;

(2) $\pi_k(O) = \pi_{k+4}(SP)$;

(3) $\pi_k(SP) = \pi_{k+4}(O) \quad (k = 0, 1, 2, \cdots)$.

Here, O denotes the direct limit of the system $\cdots \to O(n) \to O(n+1) \to \cdots$ and U, SP are defined similarly; $O(n)$, $U(n)$, $Sp(n)$ are the classical orthogonal, unitary, and symplectic groups.

The proof uses Morse theory [Bott, Bull. Soc. Math. France, **84**, (1956), 251–281; MR **19**, 291; Bott and Samelson, preceding review].

Let K be a compact Lie group, and s a geodesic segment on K from the identity e to a central point $w \in K$. Let $\Omega_s K$ be the component of s in the space of paths on K joining e and w, K_s the centralizer of s, and K^s the space of left cosets K/K_s. The formula $x \to xsx^{-1}$, $x \in K$ defines a map $f_s: K^s \to \Omega_s K$. If s contains no conjugate point of e in its interior, then $f_s^*: H^*(\Omega_s K; z) \to H^*(K^s; z)$ is an epimorphism. Using this preposition and earlier results of his, the author shows that $f_s: U(2n)/U(n) \times U(n) \to \Omega_s U(2n)$ is a $2n$-equivalence. Since $U(2n)/U(n) \times U(n)$ has the same $2n$-type as the universal base space of $U(n)$, $\Omega^{-1}U(n)$, it follows that $\Omega^{-1}U$ and $\Omega_s U$ have isomorphic homotopy groups; hence (1).

The results for O and SP are proved in a similar but more complicated manner.

In the meantime, a detailed proof of (1), different from the one sketched here, has appeared [the review follows]. *J.-P. Meyer* (Baltimore, Md.)

1589:

Bott, Raoul. The space of loops on a Lie group. Michigan Math. J. **5** (1958), 35–61.

The importance of the author's researches on the application of Morse theory to the study of Lie groups is generally recognised; many workers have been glad to apply his results. This paper is an integral part of these researches. Let K be a connected, compact Lie group, and let $\Omega' K$ be the identity-component of the loop-space on K. Then this paper shows how to calculate the Hopf algebras $H_*(\Omega' K)$ and $H^*(\Omega' K)$ in terms of properties of K already studied by the classification theory for Lie groups. The proof is based on an earlier paper by R. Bott and H. Samelson [Amer. J. Math. **80** (1958), 964–1029]; we shall refer to this paper as AMT. In turn, the present paper provides (in § 8) a proof, different from the original one, of the author's celebrated results on the homotopy

groups of unitary groups: If $k<2n$, then $\pi_k(U(n))$ is Z when k is odd, 0 when k is even; the first unstable group is given by $\pi_{2n}(U(n))=Z_{n!}$. [See #1588 above and #1590 below.]

The author uses the following construction. Let s be a circle on K, that is, a homomorphism $s:S^1\to K$. Let K_s be the centraliser of Im(s), and let K^s be the coset space K/K_s. Then a map

$$g^s:K^s\to\Omega' K$$

is defined by

$$(g^s(xK_s))(t)=xs(t)x^{-1}s(t)^{-1}\quad (x\in K,\ t\in S^1).$$

The author shows (Theorem 1) that the image of

$$g_*{}^s:H_*(K^s)\to H_*(\Omega' K)$$

generates the Pontryagin ring $H_*(\Omega' K)$, provided that s is a "generating circle for K".

This last phrase is defined as follows. Let T be a maximal torus containing Im(s); let $W(K,T)$ be the corresponding Weyl group; let Λ_s be the submodule of $H_1(T)$ generated by

$$\mathrm{Im}\,(s_*:H_1(S^1)\to H_1(T))$$

and all its transforms under $W(K,T)$. Let $\Sigma(K,T)$ be the set of roots of K, interpreted as homomorphisms from $H_1(T)$ to Z. Then the circle s is called a "generating circle for K" if each root θ in $\Sigma(K,T)$ takes the value 1 on some element of Λ_s.

The author shows (Theorem 2) that if K is a compact Lie group with trivial centre (a case which is sufficiently general) then K has a generating circle. This is proved by an appeal to the details of the classification theory; this work (§ 5) also produces particular generating circles for which K^s is as "small" as possible.

The proof of Theorem 1 (which occupies §§ 2, 3, 4) may be sketched as follows. Let F be the universal cover of the maximal torus T, and let $\eta:F\to T$ be the projection. Let c be a closed polygon in F, beginning and ending at the origin, and such that each vertex lies on a singular plane. Then one may construct a subvariety of $\Omega' K$ (possibly singular) by allowing suitable classes of elements in K to operate on the sides of $\eta(c)$. It was shown in AMT that in this way one can obtain a base for $H_*(\Omega' K)$. One now remarks that this construction is not affected (up to homotopy) if one substitutes a polygonal arc for each side of c, so obtaining a polygon c'. By the definition of a "generating circle" s, one can assume that each side c_α' of c' projects under η to a circle s_α which is conjugate either to s or to s^{-1}. In this case, it is possible to replace the required subvariety of $\Omega' K$ by the Cartesian product of the K^{s_α}. This leads almost directly to Theorem 1.

Granted that Theorem 1 provides a set of generators for $H_*(\Omega' K)$, one wishes to know the relations between them. Let B^* be the submodule of primitive elements in $H^*(\Omega' K)$, and write $B_s{}^*=(g^s)^*B^*\subset H^*(K^s)$. Then the author shows (Theorem 3 and § 6) that the ring $H^*(K^s)$ and its summand $B_s{}^*$ determine the Hopf algebras $H_*(\Omega' K)$ and $H^*(\Omega' K)$. Since (§ 7) the methods of A. Borel and J. Leray enable the author to handle $H^*(K^s;Q)$ and $B_s{}^*\otimes Q$, the theory is complete.

Of course, the actual calculations in a specific case may be complicated. However, in §§ 8, 9, 10 the author shows how his method gives a complete treatment in the case when K is a unitary group or an orthogonal group. The paper concludes (§ 11) with results on the symplectic groups and the exceptional group G_2.

{One misprint may be slightly confusing; at the beginning of § 11, the reference to "Theorem 6" should refer to Theorem 4.} *J. F. Adams* (Cambridge, England)

1590:

Bott, R.; and Milnor, J. On the parallelizability of the spheres. Bull. Amer. Math. Soc. **64** (1958), 87–89.

In this short note, Bott sketches a proof of the following theorem: For any fibre bundle with the sphere S^{4k} as base and the orthogonal group O_m as structure group, the kth integral Pontrjagin class, $p_k\in H^{4k}(S^{4k},Z)$, is divisible by $(2k-1)!$. The proof is based on results announced by Bott [#1588].

Milnor then uses this result to prove the following theorem: There exists an O_m-bundle over S^n such that the mod 2 Stiefel-Whitney class $w_n\neq 0$ if and only if $n=1, 2, 4$, or 8. This theorem has the following two corollaries: (1) The sphere S^r is parallelizable only for $r=1$, 3, or 7. (2) There exists a division algebra of rank n over the real numbers only for $n=1, 2, 4$, or 8 (associativity is not postulated). These two corollaries provide the final solution to a couple of famous problems. Bott mentions that M. Kervaire has also obtained these two corollaries.

In deriving this theorem from Bott's result, Milnor makes essential use of the following two theorems of W. T. Wu: (a) Using cup products and Steenrod squares, it is possible to express the mod 2 Stiefel-Whitney classes w_i of any O_m-bundle in terms of those Stiefel-Whitney classes w_k such that k is a power of 2. In particular, the Stiefel-Whitney classes of any O_m-bundle over S^n vanish if n is not a power of 2 [see Wu, C. R. Acad. Sci. Paris **230** (1950), 918–920; MR **12**, 42]. (b) For any O_m-bundle, the Pontrjagin classes reduced mod 4 are completely determined by the Stiefel-Whitney classes mod 2 [Wu, Acta Math. Sinica **4** (1954), 323–346; MR **18**, 225]. Actually Wu's complete formulas are not needed; only the fact that if $w_i=0$ for $0<i<4k$, then the kth Pontrjagin class p_k reduced mod 4 is the image of the Stiefel-Whitney class w_{4k}, is needed.

The derivation of Milnor's theorem from Bott's theorem and these two theorems of Wu is surprisingly easy.

W. S. Massey (Providence, R.I.)

1591:

Milnor, John. Some consequences of a theorem of Bott. Ann. of Math. (2) **68** (1958), 444–449.

This paper contains full proofs of some results announced in the paper reviewed above.

W. S. Massey (Providence, R.I.)

1592:

Kervaire, Michel A. On the Pontryagin classes of certain SO(n)-bundles over manifolds. Amer. J. Math. **80** (1958), 632–638.

The main result of this paper may be stated as follows: Let M be a compact orientable manifold of dimension $4s$ and let $p:E\to M$ be a principal fibre bundle over M with group SO(n) (the group of all $n\times n$ real orthogonal matrices of determinant $+1$). Assume that $4s<n$, and that the bundle admits a cross-section f over $M-U$, where U is a small spherical neighborhood of some point of M. Then the obstruction to extending f to a cross-section over all of M may be considered to be an element of the group $\pi_{4s-1}(\mathrm{SO}(n))$, which is infinite cyclic; suppose

the obstruction is λ times a generator of $\pi_{4s-1}(\mathrm{SO}(n))$, where λ is an integer. Theorem: The $4s$-dimensional Pontrjagin class of the bundle is $\pm a_s \cdot \lambda \cdot (2s-1)!$ times the generator of the integral cohomology group $H^{4s}(M)$, where $a_s = 1$ or 2 according as s is even or odd.

In the proof of this theorem, essential use is made of recent results of R. Bott on the stable homotopy groups of the classical groups [#1588 above]. The author shows how the case $s=1$ of this theorem implies the following result of V. Rohlin: If the 2nd Stiefel-Whitney class of a compact orientable 4-manifold vanishes, the Pontrjagin class is divisible by 48 [Dokl. Akad. Nauk USSR (N.S.) **81** (1951), 19–22; MR **13**, 674].

The author also proves analogous theorems for the case of principal unitary bundles and principal symplectic bundles over a manifold M which admit cross-sections over $M-U$ as above. *W. S. Massey* (Providence, R.I.)

1593:

Peterson, Franklin P. Some remarks on Chern classes. Ann. of Math. (2) **69** (1959), 414–420.

In this paper, the author studies the problem of classifying complex n-plane bundles over CW-complexes of dimension $2n$ or less. This is equivalent to studying bundles whose group is $\mathrm{U}(\infty)$, the union of the finite unitary groups $\mathrm{U}(n)$. He is able to quote the homotopy groups of $\mathrm{U}(\infty)$ from Bott [#1588 above]; he proceeds to study the Postnikov invariants k^{2j+1} of $\mathrm{U}(\infty)$; he finds that k^{2j+1} is an element of order $(j-1)!$. This leads to the following useful result. Suppose that ξ is an n-plane bundle over M, that $\dim M \leqq 2n$, and that the only torsion in $H^{2j}(M)$ is prime to $(j-1)!$. Then ξ is a product bundle if and only if the Chern classes $c_1(\xi), \cdots, c_n(\xi)$ are all zero. *J. F. Adams* (Cambridge, England)

1594:

Milnor, John. On the Whitehead homomorphism J. Bull. Amer. Math. Soc. **64** (1958), 79–82.

The author considers the Whitehead homomorphism $J: \pi_{r-1}(\mathrm{SO}_n) \to \pi_{n+r-1}(S^n)$ in the stable range and, using the work of Borel, Hirzebruch and von Staudt, proves the following theorem. Let q be an odd prime and let r be any multiple of $2(q-1)q^i$, $i \geqq 0$. Then for $n>r$ the image $J\pi_{r-1}(\mathrm{SO}_n) \subset \pi_{n+r-1}(S^n)$ contains a cyclic subgroup of order q^i. The proof is based on the following geometric theorem. Let ξ be a SO_n bundle over S^r corresponding to an element $\lambda \in \pi_{r-1}(\mathrm{SO}_n)$. If $J\lambda = 0$ then there exists an oriented manifold M^r, differentiably imbedded in the sphere S^{n+r}, and having the following property: Some map $q: M^r \to S^r$ of degree $+1$ is covered by a bundle map of the normal bundle of M^r into the given bundle ξ.

Applying the Hirzebruch formulae for the index of such a manifold, and using the bundles constructed by Borel and Hirzebruch, one can estimate the order of $J\lambda_0$ for a particular λ_0. It is at this point that the arithmetical work of von Staudt is applied. *R. Bott* (Cambridge, Mass.)

1595:

Kervaire, Michel A. An interpretation of G. Whitehead's generalization of H. Hopf's invariant. Ann. of Math. (2) **69** (1959), 345–365.

In Comment. Math. Helv. **28** (1954), 17–86 [MR **15**, 890], Thom gives a construction which assigns to a submanifold V of a manifold M and a field of n-frames orthogonal to V in M a map of M into S^n. Furthermore, he shows that any map of M into S^n is homotopic to a map obtained by this construction. This construction and theorem are used throughout the present paper.

The main construction of this paper is a generalization of the Hopf construction. Given regular imbeddings of manifolds M^p in E^{p+u} and M^q in E^{q+v}, a field of u-frames F_u and a field of v-frames F_v orthogonal to M^p and M^q in E^{p+u} and E^{q+v} respectively, and a homotopy class of maps φ of $M^p \times M^q$ into S^m, one obtains an element $G\varphi \in \pi_{p+q+u+v}(S^{m+u+v})$ as follows: From φ one obtains a submanifold V of $M^p \times M^q$ and a field of m-frames F_m orthogonal to V. Combining F_m, F_u, and F_v one obtains a field of $(m+u+v)$-frames orthogonal to V in $E^{p+q+u+v}$ which is then used to define a map of $S^{p+q+u+v}$ into S^{m+u+v}.

Suppose $f: S^{d+n+1} \to S^{n+1}$ is a map of class C^{d+n+1}. Removing a point from S^{d+n+1} one obtains a map $f': E^{d+n+1} \to S^{n+1}$. Let q and q' be regular values of f' and M and M' the inverse images of q and q'. Let F and F' be fields of $n+1$-frames orthogonal to M and M' in E^{d+n+1} formed by taking the inverse images of fixed frames at q and q'. Let $\varphi: M \times M' \to S^{d+n}$ be given by $\varphi(x, y) = (y-x)/|y-x|$, treating the elements of E^{d+n+1} as vectors. Let $h(f) = G\varphi$. $h(f)$ is shown to depend only on the homotopy class of f and to define a homomorphism of $\pi_{d+n+1}(S^{n+1})$ into $\pi_{2d+2n+2}(S^{d+3n+2})$ which coincides to within sign and stable suspension with the generalization of the Hopf invariant defined by G. Whitehead [Ann. of Math. (2) **51** (1950), 192–237; MR **12**, 847] where the latter is defined, $d < 2n-1$.

Using the construction G a notion of linking coefficient is also defined. As an application of h it is shown that a regular imbedding of S^d in E^{n+d} induces a trivial normal bundle on S^d if $2n > d+1$.

{Paragraph three in the introduction to this paper is rather misleading.} *E. H. Brown* (Waltham, Mass.)

1596:

James, I. M. Spaces associated with Stiefel manifolds. Proc. London Math. Soc. (3) **9** (1959), 115–140.

The spaces referred to in the title are the projective spaces, quasi-projective spaces, stunted projective spaces, and stunted quasi-projective spaces over the real numbers, complex numbers, or quaternions. The quasi-projective spaces are defined simultaneously for all three cases. It turns out that in the real case the projective and quasi-projective spaces are the same, and in the complex case the quasi-projective space is homeomorphic to the suspension of the projective space with the north and south poles identified {unfortunately, the author neglects to mention the necessity for identifying the poles of the suspension}. In the quaternionic case, no such simple relation holds; however, the quaternionic quasi-projective space admits a decomposition as a cell complex with a single vertex and cells of dimension 3, 7, 11, etc. A stunted projective space is obtained by identifying a projective subspace to a point. Similarly, a stunted quasi-projective space is obtained by starting with a quasi-projective space and then identifying a quasi-projective subspace to a point.

The main theorems of this paper are concerned with the S-type of these spaces (two spaces belong to the same S-type if there exist numbers i and j such that the i-fold suspension of one space belongs to the same homotopy

type as the j-fold suspension of the other). Also, the author shows how the S-type classification of these spaces is related to the question of the existence of cross sections of certain Stiefel manifolds (considered as fibre spaces over spheres). *W. S. Massey* (Providence, R.I.)

1597:

Kuratowski, K. Cohomotopic multiplication and duality theorems concerning arbitrary subsets of Euclidean space. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 753–758.

This paper contains statements (without proofs) of the author's results on duality. For $n \geq 2$ let $\mathfrak{E}^n$, $\mathfrak{S}_n$, $\mathfrak{P}_n$ denote, respectively, n-dimensional euclidean space, the n-sphere $\mathfrak{E}^n \cup (\infty)$, and the space $\mathfrak{E}^n - (0)$. $\mathfrak{P}_n{}^X$ will denote the space of continuous maps of X into $\mathfrak{P}_n$ with the compact-open topology. The author studies the space $\mathfrak{P}_n{}^X$ for the case when X is a subspace of $\mathfrak{E}^n$. It is stated that if p, $q \in \mathfrak{E}^n - X$ then p, q belong to the same quasi-component of $\mathfrak{S}_n - X$ if and only if the functions $x-p$, $x-q$ belong to the same component of $\mathfrak{P}_n{}^X$.

Let $\mathfrak{L}(\mathfrak{P}_n{}^X)$ denote the space of components of $\mathfrak{P}_n{}^X$. This can be given a group structure by means of cohomotopy multiplication so that it becomes a topological group. Let $\infty \in Y \subset \mathfrak{S}_n$ and let $\mathfrak{N}(Y)$ denote the topological group of integer valued normed measures defined on closed-open subsets of Y. The main duality theorem asserts that for each $X \subset \mathfrak{E}^n$ there exists a (homeomorphic) isomorphism μ of $\mathfrak{L}(\mathfrak{P}_n{}^X)$ into $\mathfrak{N}(\mathfrak{S}_n - X)$. If X is locally compact, μ is onto. *E. H. Spanier* (Berkeley, Calif.)

1598:

Švarc, A. S. The genus of a fiber space. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 219–222. (Russian)

Working in the category of locally trivial fibre spaces over normal spaces, the author introduces "the genus ('rod') of a fibre space", $g(E, B, F, p)$, equal to the least number of open sets covering the base over each of which there is a cross-section. This generalizes an earlier definition [Švarc, Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4 (76), 209–214; MR **19**, 668]. Several straightforward properties of the genus are stated, including the following. Theorem 1: $g(E, B, F, p) \leq \operatorname{cat} B$, and equality holds if E is contractible. Theorem 2: If the fibre is $(s-1)$-connected and B is a k-dimensional polyhedron, then

$$g(E, B, F, p) < (k+1)(s+1)^{-1}+1.$$

Using cohomology, he obtains theorem 4: If there are classes $x_1, \cdots, x_n \in H^*(B; \pi)$ such that $p^*x_i = 0$ and $x_1, x_2, \cdots, x_n \neq 0$, then $g(E, B, F, p) \geq n+1$. For principal fibrings, a new light is cast on the Milnor construction for classifying spaces; the partial spaces $B_n(G)$ are classifying for G-principal fibre spaces of genus $\leq n$. Such proofs as are given in this note are brief and straightforward. *J. Stasheff* (Oxford)

1599:

Homma, Tatsuo. On Dehn's lemma for S^3. Yokohama Math. J. **5** (1957), 223–244.

The author offers a proof of Dehn's lemma for the special case of the 3-sphere. In the meantime Dehn's lemma has been proved in full generality by Papakyriakopoulos [Ann. of Math. (2) **66** (1957), 1–26; MR **19**, 761]. *R. H. Fox* (Princeton, N.J.)

1600:

Papakyriakopoulos, C. D. Some problems on 3-dimensional manifolds. Bull. Amer. Math. Soc. **64** (1958), 317–335.

This is a lucid account of the recent phenomenal development of the topology of 3-dimensional manifolds. First the sphere theorem, Dehn's lemma and the loop theorem are explained and shown to be variations on the same central theme. Next comes a comparative study of their proofs, and a sketch of some of their applications and generalizations. Next the present state of the classification problem for orientable closed 3-manifolds is considered, with special reference to Milnor's recent results and to the crucial position of the sphere theorem, Dehn's lemma and Poincaré's conjecture. The Heegard diagrams, the classification of the lens spaces and Kneser's conjecture are discussed. All of this adds up to an indication of the probable future development of the problem. Finally it is shown by several remarks that the classification problem for open 3-manifolds is going to be a great deal harder than it will be for the compact ones.

{The reviewer would like to add that the principal development in the interval since the delivery of this address was the proving of Kneser's conjecture by J. Stallings [Dissertation, Princeton, 1959], and also by Milnor and Kraus [Kraus, Senior thesis, Princeton, 1959], and, it has been reported, by David Epstein in England.} *R. H. Fox* (Princeton, N.J.)

1601:

Poénaru, Valentin. Sur les variétés simplement connexes, compactes à trois dimensions. C. R. Acad. Sci. Paris **247** (1958), 624–626.

A triangulated simply connected compact connected 3-dimensional manifold K is subjected to a sequence of combinatorial operations. *R. H. Fox* (Princeton, N.J.)

1602:

Poénaru, Valentin. Sur les variétés simplement connexes, compactes à trois dimensions. C. R. Acad. Sci. Paris **247** (1958), 1818–1820.

The author claims that by using the operations described in the paper reviewed above he can prove that the complement of a point in K, when multiplied by euclidean 2-space, becomes euclidean 5-space. *R. H. Fox* (Princeton, N.J.)

1603:

Curtis, M. L.; and Wilder, R. L. The existence of certain types of manifolds. Trans. Amer. Math. Soc. **91** (1959), 152–160.

The authors first show that there exist polyhedral 3-manifolds in the 4-sphere S^4, which are homology spheres over the integers, but whose fundamental group is nonzero. In passing, they also prove that if G is a finitely generated abelian group, then a necessary and sufficient condition for G to be $H_1(M)$ of some orientable polyhedral 3-manifold $M \subset S^4$, is that the torsion subgroup of G be a direct sum of two isomorphic groups. Next it is shown that

there exist compact generalised 3-manifolds in S^4 which are not locally euclidean. Here, several interesting examples are given, using the wild arcs of Artin-Fox; and a curious lemma of Klee is shown to imply that an arc, wild in 3-space, is tame if considered as lying in 4-space. The third part of the paper concerns the fundamental problem of characterising the locally euclidean spaces in some manner as, e.g., Wilder's result that a separable generalised 2-manifold is locally euclidean. It was conjectured by the reviewer [Michigan Math. J. **2** (1953), 61–89; MR **16**, 159; p. 85] that a homotopy manifold (in the sense there defined) would be locally euclidean. The authors disprove the conjecture, by showing that a certain space B is a homotopy 3-manifold with the homotopy of the 3-sphere S^3, but that B is not locally euclidean. The space B is the remarkable "dog-bone" space (compactified at infinity), constructed by Bing [Ann. of Math. (2) **65** (1957), 484–500; MR **19**, 1187], which seems to be a rich source for counter-examples. {It is proved, e.g., by Bing [Bull. Amer. Math. Soc. **63** (1958), 82–84; MR **20** #3514] that $B\times S^1$ is homeomorphic to $S^3\times S^1$; and by Curtis [Duke Math. J. **24** (1957), 349–351; MR **19**, 1188] that B is embeddable in S^4.}
H. B. Griffiths (Bristol)

1604:

Conner, P. E.; and Floyd, E. E. A characterization of generalized manifolds. Michigan Math. J. **6** (1959), 33–43.

Let X be a locally compact Hausdorff space, let $x\in X$, and let K be a commutative ring, not zero. Using Alexander-Spanier cohomology with coefficients in K, the authors define the local (graded) cohomology module $H_x^*(X, K)$ by a process dual to the classical procedure of Alexandroff-Čech for the local homology groups. A sheaf (say S) is then defined over X, whose stalks are these modules $H_x^*(X, K)$, and the paper is concerned with the case when S is locally constant. For example, X is then locally connected; and X has property Q of P. A. Smith, and conversely. Next, let K be a field. Then $H_x^n(X, K)$ is the dual space of the stalk $H_n(\mathfrak{F}_x)$ of a sheaf $H(\mathfrak{F})$ defined by Borel [Michigan Math. J. **4** (1957) 227–239; MR **20** #4842], and local constancy of S is equivalent to that of $H(\mathfrak{F})$; it is equivalent also, if $\dim_K X<\infty$, to property P of Smith. Using the cited work of Borel and a spectral sequence argument, the main theorem is proved, that if $\dim_K X<\infty$ and S is locally constant, then there is an integer $n>0$ such that $H_x^i(X, K)=\delta_{in}\cdot K$ (Kronecker delta), for all $x\in X$; hence X is a locally orientable n-gm, in the sense of Wilder. This is a functorial restatement of, and answer to, a conjecture of Alexandroff that if the local Betti numbers $p^q(x)$ $(=\dim H_x^q(X, K))$ are everywhere constant, then there is an integer $n\geqq 0$ such that for all x, $p^q(x)=\delta_{qn}$; when $\dim X=1$, Alexandroff solved the problem affirmatively [see Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloq. Publ., New York, 1949; MR **10**, 614; p. 272]. The authors finally consider $K=Z$, the integers, and show that local constancy of S over Z implies it over Z_p, p any prime. From this, they deduce the statement of the main theorem with $K=Z$, provided that $H_x^i(X, Z)$ is finitely generated for each i (e.g., if X is cohomologically locally connected in each dimension). {In the statements of Theorems 6.1 and 7.3, third and fifth lines respectively, read "... U of x" and "... all $y\in U$", respectively for "U of y" and "$x\in X$".}
H. B. Griffiths (Bristol)

1605:

Kinoshita, Shin'ichi. Alexander polynomials as isotopy invariants. I. Osaka Math. J. **10** (1958), 263–271.

Let K be an n-dimensional complex in $(n+2)$-dimensional space S and let $\kappa_1, \cdots, \kappa_\mu$ be a basis for the free abelian group $H_n(K)$. Let $t_1, \cdots, t_\mu$ be the corresponding dual basis of $B_1(S-K)$ and let $M(t_1, \cdots, t_\mu)$ be a Jacobian matrix at the canonical homomorphism $\pi(S-K)\to B_1(S-K)$. The author calls the greatest common divisor $\Delta^{(d)}(t_1, \cdots, t_\mu)$ of the dth elementary ideal of $M(t_1, \cdots, t_\mu)$ the dth Alexander polynomial of K in S. Also let $M(t)$ be the matrix obtained from $M(t_1, \cdots, t_\mu)$ by setting $t_1=\cdots=t_\mu=t$ and denote the g.c.d. of the dth elementary ideal of $M(t)$ by $\Delta^{(d)}(t)$. The polynomials $\Delta^{(d)}(t_1, \cdots, t_\mu)$ and $\Delta^{(d)}(t)$ $(d=1, 2, \cdots)$ are invariants of type of placement of K in S and choice of basis $\kappa_1, \cdots, \kappa_\mu$.

In order to calculate these polynomials the author generalizes the Dehn presentation of a knot group to a presentation of $\pi(S-K)$. Using Artin's method of generating 2-spheres in 4-space by "rotating" knots, it is observed that given any knot polynomial and integer $n\geqq 1$ there is an n-dimensional manifold K in $(n+2)$-space whose polynomial $\Delta^{(1)}(t)$ is the given knot polynomial. {But the reviewer would like to note that there is a 2-sphere in 4-space whose polynomial $\Delta^{(1)}(t)$ is not symmetric and is therefore not the polynomial of any knot.} In the final section the polynomial $\Delta^{(2)}(t)$ is used to show that a certain pair of imbeddings in 3-space of a theta curve are not of the same type.
R. H. Fox (Princeton, N.J.)

1606:

Hosokawa, Fujitsugu. On ∇-polynomials of links. Osaka Math. J. **10** (1958), 273–282.

If $\Delta(t_1, \cdots, t_\mu)$ denotes the Alexander polynomial of a link of multiplicity μ [R. H. Fox, Ann. of Math. (2) **59** (1954), 196–210; MR **15**, 931] then for $\mu\geqq 2$, $\Delta(t, \cdots, t)$ is divisible by $(1-t)^{\mu-2}$. The author defines

$$\nabla(t)=\begin{cases}\Delta(t, \cdots, t)/(1-t)^{\mu-2} & \text{for } \mu\geqq 2,\\ \Delta(t) & \text{for } \mu=1.\end{cases}$$

Theorem 1: $\nabla(t)$ is a symmetric polynomial of even degree (unless $\nabla(t)=0$). Theorem 2: $\pm\nabla(1)$ is equal to the minor determinant of order $\mu-1$ of the matrix

$$\begin{pmatrix} -\sum_{k=1}^{\mu} l_{1,k} & l_{1,2} & \cdots & l_{1,\mu}\\ l_{2,1} & -\sum_{k=1}^{\mu} l_{2,k} & \cdots & l_{2,\mu}\\ l_{\mu,1} & l_{\mu-1,1} & \cdots & -\sum_{k=1}^{\mu} l_{\mu,k}\end{pmatrix},$$

where $l_{i,i}=0$ and $l_{i,j}$ $(i\neq j)$ denotes the linking number of the ith and the jth components of the link. Theorem 3: Given $\mu\geqq 1$ and a symmetric polynomial $f(t)$ of even degree, there is a link of multiplicity μ for which $\nabla(t)=f(t)$.
R. H. Fox (Princeton, N.J.)

1607:

Hashizume, Yoko. On the uniqueness of the decomposition of a link. Osaka Math. J. **10** (1958), 283–300.

The results of H. Schubert [S.-B. Heidelberger Akad. Wiss. Math. Nat. Kl. **1949**, no. 3, 57–104; MR **11**, 196] are generalized from (oriented) knots to (oriented) links. An oriented link l of multiplicity n is the union of n disjoint simple closed oriented polygons $k^1\cup k^2\cup\cdots\cup k^n$. Let k^j

be the union of oriented arcs u and v and let Q be a 3-cell such that

$$Q\cap l = k^1\cup k^2\cup\cdots\cup k^{i-1}\cup u,$$
$$\overline{S^3-Q}\cap l = v\cup k^{i+1}\cup\cdots\cup k^n.$$

Then there is an oriented arc w on the boundary of Q such that $l_1=k^1\cup\cdots\cup k^{i-1}\cup(u\cup w)$ is an oriented link of multiplicity i and $l_2=(w^{-1}\cup v)\cup k^{i+1}\cup\cdots\cup k^n$ is an oriented link of multiplicity $n-i+1$. The link l is then said to be decomposed into l_1 and l_2, or l is said to be a product of l_1 and l_2 (but product is not a unique operation unless $i=n-i+1=n=1$). A link is called trivial if it lies on a 2-sphere. A decomposition of l into l_1 and l_2 is called trivial if either l_1 or l_2 is a trivial knot (and the other one is therefore of the same type as l). A link l is called separable if there is a 2-sphere in S^3-l whose interior and exterior both intersect l. A link is called prime if it is non-trivial and non-separable and admits only trivial decompositions. Main theorem: Every non-trivial, non-separable link decomposes uniquely into prime links. The genus $g(l)$ of an oriented link l is the minimum of the genera of the connected orientable non-singular surfaces in S^3 that span l. Theorem 2: If l decomposes into l_1 and l_2 then $g(l)=g(l_1)+g(l_2)$. *R. H. Fox* (Princeton, N.J.)

1608:

★**Berge, Claude. Théorie des graphes et ses applications.** Collection Universitaire de Mathématiques, II. Dunod, Paris, 1958. viii+277 pp. 3400 francs.

This is the second book on graph theory ever written. The previous book is the already classical: D. König, *Theorie der endlichen und unendlichen Graphen* [Akademische Verlag., Leipzig, 1936; Chelsea, New York, 1950; MR **12**, 195]. There are however several books on combinatorial analysis and topology which contain a chapter on graph theory. There has recently been a resurgence of interest in both the theory and application of graphs, whence the author obtains the title of his book. He notes that graphs are enjoying current usage in many different disciplines, "sociogrammes (psychologie), simplexes (topologie), circuits électriques (physique), diagrammes d'organisation (économie), réseaux de communications, arbres généalogiques, etc."

Graph theory is a mixture of combinatorial analysis and 1-dimensional combinatorial topology. The author has done some excellent original research in graph theory and properly includes it in the book. In addition, several of his proofs of well known results are more concise and more elegant than previous proofs appearing in the literature. The author is to be congratulated on his skillful interpolation of examples from several different fields and on his effective use of diagrams as an assistance in understanding proofs, e.g., on p. 99. The printing of the text and the figures is beautiful.

We now summarize the Table of Contents: Chapter 1 Definitions, 2 Descendance, 3 Functions on an infinite graph, 4 Fundamental numbers, 5 Solutions of a graph, 6 Games on a graph, 7 The shortest path, 8 Transportation networks, 9 Semidegrees, 10 Covers of a simple graph, 11 Factors, 12 Centrality, 13 Diameter, 14 Matrices associated with a graph, 15 Incidence matrix, 16 Trees, 17 Euler's problem, 18 Covers of an arbitrary graph, 19 Semifactors, 20 Connectivity, 21 Planar graphs; Appendix I Game theory, II Transportation problems, III The use of potential for transportation networks, IV Unsolved problems and conjectures, V "Notice sur quelques principes fondamentaux d'énumération, par J. Riguet".

The bibliography is arranged separately for each chapter; unfortunately, there is no author index. The organization of the material is unorthodox and is not optimally arranged logically, particularly since the basic concept of a tree is not introduced until Chapter 16.

We note the following few misprints: (1) In the Table of Contents, section 3 of Chapter 1, read "cycles" for "circuits"; (2) p. 275: for "R. M. Forster" read "R. M. Foster"; (3) p. 275: for "Davis" read "R. L. Davis"; and (4) in reference [2] for Appendix V, the reviewer is listed as one of four authors of a joint work written by the other three authors.

Appendix V, which was written by J. Riguet, consists essentially of a translation from German into French of parts of Pólya's basic work on enumeration problems. However Pólya is only given credit for one minor definition. In addition, the notation of Appendix V is not clear and the reader who is interested in learning enumeration techniques would do much better to refer to the original paper of Pólya or the recent book of Riordan on combinatorial analysis. The first three appendices on the other hand are interesting and worthy; they contain a survey of the interaction between game theory, graph theory, and linear programming. Appendix IV with its list of 14 unsolved problems is one of the most interesting features of the book. However the last three of these problems appear hopeless at present since they are variations of the 4 color conjecture. On the other hand, problem 11, which asks for the maximum connectivity of a p, q graph (one with p points and q lines) is relatively trivial: the answer is $[2q/p]$, 1, or 0 when $q>$, $=$, or $<p-1$ respectively, a result which can be readily proved. This problem also asks for the minimum diameter of a p, q graph: the answer is 1, 2, or ∞ depending on the value of q in terms of p.

On p. 139 there is a theorem attributed to Heller, Tompkins and Gale, which is also due to A. J. Hoffman. There are several theorems which are somewhat incongrously attributed to joint authorship such as the theorems of König-Ore, Birkhoff-von Neumann, Poincaré-Veblen-Alexander, Tutte-Bott, and von Neumann-Nash. On p. 109, König is credited with the theorem to the effect that every complete oriented graph is hamiltonian; however in his book, König attributes this theorem to Rédei and states the generalization (also due to Rédei, but not mentioned in the book under review) that the number of complete cycles in such a graph is odd. Although the author is generous in assigning names to theorems, there is an oversight in not attributing to H. J. Ryser the very beautiful theorem on p. 86 characterizing the existence of a 0, 1 matrix with prescribed row and column sums.

The book contains a considerable number of new results on graph theory which have been discovered since the book of König, and is therefore a most welcome addition to the mathematical literature. However, it is unfortunate that the author decided to restrict the length of the book by omitting several fundamental theorems, e.g. the theorem of Frucht that any abstract finite group is isomorphic to the automorphism group of some graph, and the theorem of Whitney which exhibits a class of triangulated hamiltonian planar graphs. In summary, in general this is a good book and a welcome book; the presentation is novel and frequently ingenious. *F. Harary* (Ann Arbor, Mich.)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 1880.

1609:

Kautny, Walter. Über die durch harmonischen Umschwung erzeugbaren Strahlflächen. Monatsh. Math. **63** (1959), 169–188.

Ein harmonischer Umschwung ist eine Raumbewegung, die sich aus einer gleichförmigen Drehung um eine Achse und einer harmonischen Schwingung längs dieser Achse zusammensetzt. Eine Gerade des Raumes erzeugt eine Strahlfläche, welche für rationale Frequenz der Schwingung algebraisch ist. Die Fläche wird konstruktiv und analytisch untersucht. Verf. behandelt die Schichtenlinien, die Selbstschnitte, Eigen- und Schlagschatten bei Parallelbeleuchtung, umschriebene Böschungstorsen und die Haupttangentenkurven. *O. Bottema* (Delft)

1610:

Garsia, Adriano. On surfaces with a rectilinear geodesic circle. Ann. Mat. Pura Appl. (4) **46** (1958), 201–213.

A class of surfaces containing a straight line segment AB and a point F with constant geodesic distance from a variable point of AB is constructed; they are the envelopes of families of cones which are locally solutions. The solutions are used to obtain the rectilinear detonation front (target AB) by a single detonator F, the layer of explosive being warped as the surface giving the solution of our problem. *A. Švec* (Prague)

1611:

Svoboda, Karel. Propriétés projectives des surfaces minima à circonférences de courbure normale. Časopis Pěst. Mat. **83** (1958), 287–316. (Czech. Russian and French summaries)

On appelle $\mathbf{M}$ une surface $M=M(u,v)$ plongée dans l'espace S_n à courbure constante c dont toutes les indicatrices de courbure normale d'ordre $\leqq m-1$, $2\leqq m\leqq[\frac{1}{2}n]$, sont des circonférences aux centres au point M [v. O. Borůvka, Publ. Fac. Sci. Univ. Masaryk **165** (1932), 1–22]. Pourqu'une surface (M) de l'espace projectif P_n puisse être considérée comme une surface $\mathbf{M}$ de l'espace (1) euclidien, (2) non-euclidien S_n (S_n étant subordonné à P_n et déterminé par une quadrique absolue), il faut et il suffit qu'elle soit douée d'un réseau conjugué tel que (1) les transformations laplaciennes (M_1), (M_{-1}) soient des courbes situées avec leurs espaces osculateurs d'ordre $m-1$ sur une quadrique régulière $A\subset P_{n-1}\subset P_n$; (M_1), (M_{-1}) sont arbitraires, ou bien une, ou bien toutes les deux se trouvent plongées dans $P_{m-1}\subset P_n$; (2) $(M_{-m}),\cdots,$ (M_m) soient situées sur une quadrique régulière $A\subset P_n$ et (M_{-r}), (M_r) soient conjuguées par rapport à A à $(M_{-r+1}),$ $\cdots,(M_{r-1})$. La suite $\{(M_i)\}$ ne s'arrête pas après m transformations, ou bien elle s'arrête dans un sens ou dans les deux sens après m transformations de la manière de Goursat. *A. Švec* (Prague)

1612:

Godeaux, Lucien. Sopra una estensione della nozione di congruenze stratificabili. Boll. Un. Mat. Ital. (3) **13** (1958), 551–554. (French summary)

Sunto di una conferenza fatta nell'Istituto di Geometria a Bologna; per gli dettagli vedi Godeaux, Math. Nachr. **18** (1958), 57–63 [MR **20** #5493]. *A. Švec* (Prague)

1613:

Godeaux, Lucien. Quelques remarques sur les suites de Laplace. Bull. Soc. Math. Belg. **9** (1957), 21–30.

Les notations utilisées dans cet article sont celles que l'auteur a introduites dans son mémoire fondamental [*La théorie des surfaces et l'espace réglé*, Hermann, Paris, 1934]. M. Godeaux considère une suite de Laplace dans un espace projectif S_r à r dimensions; il montre d'abord que la projection d'une suite de Laplace L de S_r à partir d'un espace S_{r-p-1} sur un espace S_p est une suite de Laplace $\bar{L}$. L'espace à $r-p$ dimensions défini par $r-p+1$ points consécutifs de L rencontre S_p en un point J; l'auteur démontre que J appartient à une suite de Laplace inscrite dans le polyèdre à faces à $r-p$ dimensions associé à la suite $\bar{L}$ et qu'inversement toute suite de Laplace inscrite dans le polyèdre à faces à $r-p$ dimensions associé à $\bar{L}$ peut être obtenue par ce procédé. Enfin, l'auteur termine par un problème en relation directe avec la stratifiabilité des congruences. *M. Decuyper* (Lille)

1614:

Pimiä, Lauri. Die quaternionische Begründung der reellen Liniengeometrie. Soc. Sci. Fenn. Comment. Phys.-Math. **17** (1955), no. 10, 20 pp.

Teilweise im Anschluß an frühere Arbeiten des Verf. zur Kugelgeometrie [Ann. Acad. Sci. Fenn. Ser. A. I. Math.-Phys. no. 4 (1941); no. 32 (1945); MR **7**, 483; **8**, 350] wird die reelle Liniengeometrie durch eine reelle Zuordnung zwischen den Geraden des Raumes und homogenen hyperbolischen Quaternionen begründet. (Diese gehen durch eine einfache Substitution aus den Hamiltonschen Quaternionen hervor.) Jedem Punkt des projektiven R_3 entspricht eine hyperbolische Quaternion $X=x_0+x_1e_1+x_2e_2+x_3e_3$. Ist eine Gerade des R_3 durch zwei Punkte gegeben, so werden ihre Plückerschen Linienkoordinaten mit dem Quaternionenpaar verknüpft, das den gegebenen Punkten zugeordnet ist. Jeder reellen Geraden ist eineindeutig ein Punkt der projektiven quaternionischen Geraden, ein homogenes Quaternionenpaar, zugeordnet. Regelscharen zweiten Grades, Regelscharbüschel und lineare Strahlenkongruenzen, Regelscharbündel und lineare Strahlenkomplexe lassen sich durch lineare Quaternionenfunktionen darstellen. Mit Hilfe dieses Kalküls wird eine Reihe von Sätzen über diese liniengeometrischen Mannigfaltigkeiten hergeleitet. *F. Reutter* (Zbl **66**, 145)

1615:

Alekseev, N. I. On a one parameter family of stratifiable orthogonal pairs. Aviacion. Inst. Sergo Ordžonikidze. Trudy Inst. no. **61** (1956), 20–29. (Russian)

1616:

Čech, Eduard. Détermination du type différentiel d'une courbe de l'espace à deux, trois ou quatre dimensions. Czechoslovak Math. J. **7** (**82**) (1957), 599–631. (Russian summary)

Let C be a curve in E_n and t a parameter on C. Suitable

regularity assumptions are made. Let

$$\frac{dX}{dt} = K_0(t)e_1(t), \cdots, \frac{de_n}{dt} = -K_{n-1}(t)e_{n-1}(t)$$

be the Frenet formulas for C ($X(t)$=point of C), let $s=\int K_0dt$, $\sigma_i=\int K_idt$, let $T_1(t)$, $T_2(t)$, $\cdots$ be the tangent, the osculating plane, $\cdots$ of C in $X(t)$, and let cl (f) denote the differentiable class of f. The sequence (cl (X), cl (T_1), $\cdots$, cl (T_{n-1})) is called the differential class of C. cl (X) [resp. cl (T_i)] takes its maximum value when $t=s$ [$t=\sigma_i$]. The square matrix formed by the differential classes corresponding to the choices $t=s$, $t=\sigma_1$, $\cdots$ of the parameter t is called the differential type of C. The author determines all possible differential classes and all possible differential types for $n=2$, 3 and 4.

J. L. Tits (Brussels)

1617:

Čech, Eduard. Classe différentielle des courbes. Sections et projections. Rev. Math. Pures Appl. **2** (1957), 151–159.

Let (r_{ij}) be the differential type of C [cf. the preceding review]. Its diagonal (r_{00}, $\cdots$, $r_{n-1,\,n-1}$) is a projective invariant (while (r_{ij}) itself need not be one), and is called the projective type of C. A "binding" ("liaison") is a set of indices i_1, $\cdots$, i_r such that there exists a parameter t for which cl ($T_{i_j}(t)$)$=r_{i_j,i_j}$ for all j; the bindings of C are also projective invariants. The author investigates now all possible projective differential types and all bindings; he examines also the behaviour of the differential class in a projection.

J. L. Tits (Brussels)

1618:

Čech, E. Sur le type différentiel anallagmatique d'une courbe plane ou gauche. Colloq. Math. **6** (1958), 141–143.

Let $s_i(C)$ be the maximal value, for all parameters t, of the differentiable class of the osculating i-sphere of C in $X(t)$ (cf. the two preceding reviews). All possible sets of values ($s_0(C)$, $\cdots$, $s_{n-1}(C)$) are listed, without proof, for $n=2$ and 3.

J. L. Tits (Brussels)

1619:

Foster, B. L. Differentiation on manifolds without a connection. Michigan Math. J. **5** (1958), 183–190.

The author deals in modern terminology with topics which were treated some 25 years ago under the name of the absolute calculus of Pascal-Vitali [see Vitali, *Geometria nello spatio hilbertiano*, Zanichelli, Bologna, 1934; and E. Bompiani, Mem. Acad. Italia **6** (1935), 269–520]. Whereas the second partial derivatives of a function of n independent variables do not form the components of a tensor in the usual sense, the composite object formed by the first and the second derivatives together forms an absolute quantity in the sense of Vitali, or the components of a tensor of type K in the terminology of the present author. A tensor field of type K is an equivalence class of sets of real differentiable, locally defined functions (T_α, $T_{\alpha\beta}$) on the manifold with a law of transformation for change of variables involving the first and second derivatives of one set of variables with respect to the other. Dually a tensor field (S^α, $S^{\alpha\beta}$) of type K^{-1} can be defined with a corresponding law of transformation, the duality being given by the fact that $T_\alpha S^\alpha + T_{\alpha\beta}S^{\alpha\beta}$ is a real number. Ordinary tensors are described as being of type J. Theorems are proved about isomorphisms which exist between spaces formed by tensors of various types.

E. T. Davies (Southampton)

1620:

Nalli, Pia. Calcolo tensoriale ed operazioni funzionali. II. Boll. Un. Mat. Ital. (3) **12** (1957), 131–144.

Verf. führt hier den Gedankengang einer vorangegangenen Arbeit mit demselben Titel [derselbe Boll. **11** (1956), 117–122; MR **18**, 274], dem ein Tensor T_i^j zugrunde lag, diesmal von einem Tensor T_{ij}^h ausgehend durch. Es handelt sich dabei im wesentlichen um eine ausführlichere Darstellung von Entwicklungen, die schon in einer ersten Veröffentlichung über dieses Thema [ibid. **10** (1955), 135–146; MR **17**, 407] teils im Abriß mitgeteilt, teils angedeutet worden sind.

E. Schönhardt (Zbl **77**, 356)

1621:

Švec, Alois. Remarque sur le tenseur de torsion de l'espace à connexion euclidienne à trois dimensions. Časopis Pěst. Mat. **84** (1959), 46–49. (Czech. Russian and French summaries)

In this note the author gives a geometrical interpretation of the torsion tensor in a three-dimensional space with a euclidean connection.

J. J. Kohn (Waltham, Mass.)

1622:

Gheorghiu, Octavian Em. Sur un système d'équations fonctionnelles que l'on rencontre dans la théorie des objets géométriques spéciaux. Com. Acad. R. P. Romîne **8** (1958), 133–139. (Romanian. Russian and French summaries)

The author works with not purely differential geometric objects, but V. V. Wagner [Dokl. Akad. Nauk SSSR **46** (1945), 347–349; MR **7**, 265] and A. Nijenhuis [*Theory of the geometric object*, Thesis, Univ. of Amsterdam, 1952; MR **14**, 320; cf. also J. Aczél, Acta Math. Acad. Sci. Hungar. **8** (1957), 19–52; MR **19**, 677; esp. p. 26] have proved that these can always be reduced to purely differential geometric objects; a considerable part of the present paper repeats this proof in special cases. The author's point of departure seems to be (he does not give details) the following: He apparently writes the transformation formula of a linear geometric object of n components in n-dimensional space in the form

$$\bar{w}_\lambda = [C_\lambda{}^l(A_k{}^\kappa)w_l + b_\lambda(A_k{}^\kappa)]|\det A_k{}^\kappa|^c$$

$$(k, \kappa, K, l, \lambda, L=1, \cdots, n;\ A_k{}^\kappa = \frac{d\bar{x}^\kappa}{dx^k},\ B_\kappa{}^k = \frac{d\bar{x}^k}{d\bar{x}^\kappa};$$

$|\det A_k{}^\kappa|^c$ could be absorbed in $C_\lambda{}^l$, b_λ, then also the resulting functional equations would be more simple). A second transformation yields

$$\begin{aligned}\bar{\bar{w}}_L &= [C_L{}^l(B_\kappa{}^K A_k{}^\kappa)w_l + b_L(B_\kappa{}^K A_k{}^\kappa)]|\det (B_\kappa{}^K A_k{}^\kappa)|^c \\ &= \{C_L{}^\lambda(B_\mu{}^K)[C_\lambda{}^l(A_k{}^\kappa)w_l + b_\lambda(A_k{}^\kappa)]|\det A_k{}^\kappa|^c \\ &\quad + b_L(B_\mu{}^K)\}|\det B_\mu{}^K|^c\end{aligned}$$

and so the author's fundamental equations

$$(1)\qquad C_L{}^l(B_\kappa{}^K A_k{}^\kappa) = C_\lambda{}^l(A_k{}^\kappa)C_L{}^\lambda(B_\kappa{}^K),$$

$$(2)\qquad b_L(B_\kappa{}^K A_k{}^\kappa) = C_L{}^\lambda(B_\mu{}^K)b_\lambda(A_k{}^\kappa) + b_L(B_\kappa{}^K)|\det A_k{}^\kappa|^{-c}$$

follow. The author's aim is to solve these equations, but in the reviewer's opinion this solution cannot be complete, as the author asserts that (1) has only the following two measurable solutions: $C_\lambda{}^l(A_k{}^\kappa)=|\det A_k{}^\kappa|^d\delta_\lambda{}^l$ ($\delta_\lambda{}^l$ is the Kronecker-symbol) and $C_\lambda{}^l(A_k{}^\kappa)=a_\lambda{}^l$, i.e., the elements of the inverse matrix of $A_k{}^\kappa$. Now (1) has evidently a lot of further measurable (even analytic) solutions, e.g. $C_\lambda{}^l(A_k{}^\kappa)=A_\lambda{}^l|\det A_k{}^\kappa|^d, a_\lambda{}^l|\det A_k{}^\kappa|^d, \exp\{D_\lambda{}^l\log|\det A_k{}^\kappa|\}$ (also the solutions multiplied by sign det $A_k{}^\kappa$ give solutions), where $A_\lambda{}^l$ are the elements of the transposed matrix of $A_k{}^\kappa$ and $D_\lambda{}^l$ those of a constant matrix [cf., e.g., O. Perron, Math. Z. **48** (1942), 136–172; MR **5**, 30]. So the solution of (2) based upon that of (1) remains incomplete too. {As a matter of fact, the problem is very interesting and a complete solution would be desirable.}

J. Aczél (Debrecen)

1623:

Ishii, Yoshihito. On conharmonic transformations. Tensor (N.S.) 7 (1957), 73–80.

Let f be a harmonic function, that is, a function satisfying $g^{ji}\nabla_j\nabla_i f=0$ in a Riemannian space with the fundamental tensor g_{ji}, ∇_j denoting the covariant differentiation with respect to g_{ji}. The function $\bar f=e^{-\frac12(n-1)\sigma}f$ is harmonic in a Riemannian space with the fundamental tensor $\bar g_{ji}=e^{2\sigma}g_{ji}$ if and only if the function σ satisfies $g^{ji}\sigma_{ji}=0$, where

$$\sigma_{ji}=\nabla_j\sigma_i-\sigma_j\sigma_i+\tfrac12 g_{ji}\sigma^a\sigma_a,\quad \sigma_i=\nabla_i\sigma.$$

A conformal transformation $\bar g_{ji}=e^{2\sigma}g_{ji}$ satisfying this condition is called a conharmonic transformation. The author proves: (1) A necessary and sufficient condition that a conformal transformation $\bar g_{ji}=e^{2\sigma}g_{ji}$ be conharmonic is that it satisfies $\bar g_{ji}\bar K=g_{ji}K$, where K is the scalar curvature. (2) A necessary and sufficient condition that a Riemannian space V_n ($n>3$) be reduced to a flat space by a suitable conharmonic transformation is that the conharmonically invariant tensor

$$Z^h_{kji}=K^h_{kji}-(n-2)^{-1}(A_k{}^hR_{ji}-A_j{}^hR_{ki}+R_k{}^hg_{ji}-R_j{}^hg_{ki})$$

vanishes identically, where K^h_{kji} and R_{ji} are curvature tensor and Ricci tensor respectively. (3) A necessary and sufficient condition that an Einstein space be transformed into an Einstein space by a conharmonic transformation is that σ satisfies $\sigma_{ji}=0$. *K. Yano* (Tokyo)

1624:

Šum, A. I. Questions of separation of variables in the equation $\Delta_2v=0$ in spaces of constant curvature. Mat. Sb. (N.S.) 47 (89) (1959), 495–512. (Russian)

Consider, in a Riemannian space with line element $ds^2=\sum_{i,j=1}^3 g_{ij}d\rho_id\rho_j$, the Laplace equation $\Delta_2v=0$, where Δ_2 is Beltrami's second differential parameter

$$\Delta_2v\equiv\frac{1}{g^{1/2}}\sum_{i,j=1}^3\frac{\partial}{\partial\rho_i}\left(g^{1/2}g^{ij}\frac{\partial v}{\partial\rho_j}\right).$$

Definition 1: The Laplace equation is P-separable provided that there is a function $P(\rho_1,\rho_2,\rho_3)$ such that the functions $P(\rho_1,\rho_2,\rho_3)\cdot f_1(\rho_1)\cdot f_2(\rho_2)\cdot f_3(\rho_3)$ are solutions, where each of the functions f_i'/f_i, for $i=1,2,3$, contains at least two independent parameters. (If $P\equiv1$, then one speaks of the Laplace equation as being completely separable.) Definition 2: The Laplace equation is partially separable provided it has solutions of the form $f_1(\rho_1)\cdot S(\rho_2,\rho_3)$, where each of the functions f_1'/f_1, $\partial\log S/\partial\rho_2$, $\partial\log S/\partial\rho_3$ contains at least two independent parameters. The present paper is concerned with the determination (for spaces of constant curvature $K\gtreqless0$) of (1) all coordinate systems in which Laplace's equation is P-separable; and (2) all coordinate systems in which Laplace's equation is partially separable. Only the answer to (1) will be stated here. By a theorem of G. Darboux [*Leçons sur les systèmes orthogonaux et les coordonnées curvilignes*, 2nd ed., Gauthier-Villars, Paris, 1910; p. 293] there is no loss of generality in supposing that

$$ds^2=H_1{}^2d\rho_1{}^2+H_2{}^2d\rho_2{}^2+H_3{}^2d\rho_3{}^2.$$

Theorem 1: If the constant curvature K is not zero, then there are no coordinate systems (other than those in which the Laplace equation is completely separable, or 1-separable) in which the Laplace equation is P-separable. [The coordinate systems for which the Laplace equation is completely separable, when $K\neq0$, were determined by M. N. Olevskiĭ, Mat. Sb. (N.S.) **27** (**69**) (1950), 379–426; MR **12**, 415.] *J. B. Diaz* (College Park, Md.)

1625:

Kaul, S. K.; and Mishra, R. S. Generalised Veblen's identities. Tensor (N.S.) 8 (1958), 159–164.

The authors obtain formal identities satisfied by the curvature tensor of an asymmetric affine connexion which reduce to the Veblen identity

$$P^i_{jkl|m}+P^i_{ljm|k}+P^i_{mlk|j}+P^i_{kmj|l}=0$$

when the connexion is symmetric.

T. J. Willmore (Liverpool)

1626:

Nijenhuis, Albert. A note on infinitesimal holonomy groups. Nagoya Math. J. 12 (1957), 145–146.

Let $P=P(M,G,\pi)$ be a differentiable principal fibre bundle over a base manifold M with Lie structure group G and with projection π of P onto M, $H'(x)$ the infinitesimal holonomy group at a point x of P, $H^*(x)$ the local holonomy group at a point x of P. The author [Indag. Math. **15** (1953), 233–249; **16** (1954), 17–25; MR **16**, 171, 172] and H. Ozeki [Nagoya Math. J. **10** (1956), 105–123; MR **18**, 232–233] proved the theorem: If dim $H'(x)$ is constant in a neighborhood of x in P, then we have $H'(x)=H^*(x)$.

This note is to present a more direct proof of the above theorem. *K. Yano* (Tokyo)

1627:

Ambrose, W. Parallel translation of Riemannian curvature. Ann. of Math. (2) 64 (1956), 337–363.

It is shown that a complete simply connected Riemannian manifold M (of class C^∞) is determined (to within isometry) by the behavior of the Riemannian curvature under parallel translation.

Let d be the dimension of the manifold and R^d real euclidean d-space. Consider the space Z of triples (a,b,Q), where $a\in R^d$, $b\in R^d$, and Q is a linear two-dimensional subspace of R^d. For the given manifold M, with fixed initial point m and tangent basis at m, there is defined a real-valued function L on Z as follows. Identify (in the natural way) R^d with the tangent space at m. Consider the space P obtained by parallel translation of Q, first along

the geodesic segment corresponding to a (i.e., with length and direction of a), then along the segment corresponding to b. Set $L(a, b, Q) = K(P)$, where $K(P)$ is the Riemannian curvature corresponding to P. The author's fundamental theorem is that, if for two manifolds M, M' the corresponding functions L, L' coincide, then between them there exists an isometry (superposing initial points and inducing in the tangent spaces a superposition of the chosen bases).

The article exploits extensively the interpretation of differential-geometric notions from the fiber bundle point of view [see, e.g., Ambrose and Singer, Trans. Amer. Math. Soc. **75** (1953), 428–443; MR **16**, 172]. The exponential mapping of the tangent plane into the manifold is examined in detail.

The function L assumes a simple form for manifolds of constant curvature (L=const), symmetric manifolds (L independent of a and b), and manifolds with analytic connection. The author poses the following unsolved problems: (1) What functions on Z can be L functions for Riemannian manifolds? (2) What can be said as to the homeomorphism of two manifolds whose L functions are sufficiently close [see, e.g., Rauch, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 440–442; MR **14**, 1015]?

I. Z. Rozenknop (RŽMat **1958** #8245)

1628:

Ambrose, W.; and Singer, I. M. On homogeneous Riemannian manifolds. Duke Math. J. **25** (1958), 647–669.

A chaque point x de l'espace tangent M_m en un point quelconque m d'une variété riemannienne homogène M, on peut attacher une transformation linéaire antisymétrique T_x de M_m, dépendant linéairement de x, et vérifiant les relations

(A) $\quad \Delta_z R_{x,y} = -R_{T_z x,y} + R_{T_z y,x} - [R_{x,y}, T_z],$

(B) $\quad \Delta_y T_x = -T_{T_y x} + [T_y, T_x]$

(où $R_{x,y}$ désigne la courbure de M et Δ_x le symbole de dérivation covariante).

Inversement, si M est une variété riemannienne complète et simplement connexe, et s'il existe une transformation linéaire T_x satisfaisant à (A) et (B), alors M est homogène (si $T_x \equiv 0$, alors M est symétrique).

Deux démonstrations de la deuxième proposition sont données; la première consiste à construire un groupe de Lie G, agissant sur M par isométries, et dont M soit un espace quotient; la deuxième utilise les résultats antérieurs de l'un des A. [Analyse ci-dessus.] *J. Lelong* (Paris)

1629:

Hiramatu, Hitosi. Riemannian manifolds and conformal transformation groups. Tensor (N.S.) **8** (1958), 123–150.

Let M be an n-dimensional Riemannian manifold, G an effective Lie group of conformal transformations of M, a transformation φ being called conformal if it is a diffeomorphism and there exists a positive real-valued function α defined on M such that $\langle d\varphi s, d\varphi t\rangle = \alpha(m)\langle s, t\rangle$ for all s, t in T_m, where T_m is the tangent space to M at $m \in M$. The main theorems of this paper concern possible dimensions of G under various further assumptions.

Let G_m be the isotropy group of G at m, i.e., $G_m = [\varphi \in G | \varphi(m) = m]$ and H_m the subgroup of elements which are isometric at m, i.e., $H_m = [\varphi \in G_m | \alpha(\varphi, m) = 1]$; let $\alpha_\varphi = \alpha(\varphi, \cdot)$. If L is any topological group let L^0 denote its identity component. Let $\tilde{\varphi}_m$ be the natural representation of G_m on T_m, K_m the kernel of $\tilde{\varphi}_m$. The first theorem states: if $n > 2$ and $m \in M$ then the mapping σ of $K_m^0 \to T_m^*$ which carries φ into $d\alpha_\varphi$ at m, is an iso into; hence $0 \leqq \dim K_m \leqq n$.

Let $R_m^* = \sigma(K_m^0)$, R_m its dual (under the scalar product) in T_m, hence $K_m^0 \approx R_m$. Let $\tilde{G}_m = \tilde{\varphi}_m(G_m)$. Another theorem says, for $n > 2$, that if G_m^0 is isometric at m (i.e., $G_m^0 \subseteq H_m$) then $n - \dim K_m \geqq \dim G - \dim G_m$. Let $\tilde{H}_m = \tilde{\varphi}_m(H_m)$ = the rotations in $\tilde{G}_m$. Another theorem says that if $n > 4$ and $n \neq 8$ and if $\dim H_m = \frac{1}{2}(n-1)(n-2)$ holds at m then the Weyl conformal curvature tensor vanishes at m.

The above theorems are used to prove numerous theorems concerning the dimension of G; some of these are known results of Sasaki, Taub and Yano while others are new. We quote only two of these: (1) If $n > 2$ and $n \neq 4$, then it is impossible that $\frac{1}{2}n(n+1) + 2 < \dim G < \frac{1}{2}(n+1)(n+2)$. (2) If G is connected and consists of homotheties (i.e., each α_φ is constant on M) with $n > 4$ and $n \neq 8$, and if $\dim G = 2 + \frac{1}{2}n(n-1)$, then M is flat.

W. Ambrose (Cambridge, Mass.)

1630:

Dalla Volta, Vittorio. Varietà geodetiche nello spazio di Siegel-Hua. Ann. Mat. Pura Appl. (4) **46** (1958), 19–42.

The author proves as follows. In a symmetric space, every totally geodesic locally euclidean subvariety is the locus of geodesics which are trajectories corresponding to a certain one-parametric group of transvections (E. Cartan) of the entire space.—He applies it to a study of totally geodesic varieties in Siegel-Hua spaces previously made by himself [Rend. Mat. e Appl. (S) **13** (1955), 294–334; MR **18**, 934], and in particular he exhibits families of such varieties previously not obtained.

S. Bochner (Princeton, N.J.)

1631:

Teleman, C. Sur une classe d'espaces riemanniens symétriques. Rev. Math. Pures Appl. **2** (1957), 445–470.

The complex and quaternionic projective spaces, considered as real spaces, introduced by G. Vrănceanu [see Acad. R. P. Romîne Stud. Cerc. Mat. **5** (1954), 173–223; MR **16**, 624] and the present author [see Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 977–1002; MR **18**, 232], can be endowed with a symmetric riemmannian metric, such that the spaces V_{2p} and V_{4p} thus obtained can be considered as anholonomic subspaces of the spheres of $2p+1$ and $4p+3$ dimensions respectively. This, of course, is connected with the fact that the spheres S_{2p+1} and S_{4p+3} are fibre bundles having as base spaces the V_{2p} and V_{4p} respectively.

Based on these results Vrănceanu conjectured that every symmetric and closed riemannian space V_n can be defined as an anholonomic subspace of a sphere S, the metric of V_n being equal to the metric of S, induced on the anholonomic subspace.

The purpose of the present paper is to prove this conjecture. This is done by proving the theorem: If the group of motions G of a homogeneous riemannian space V_n is isomorphic to the unitary group of one of the classes A, B, C, D [see L. Pontryagin, *Topological groups*, Princeton Univ. Press, Princeton, N.J., 1939; MR **1**, 44], and if the stability group of a point of V_n is irreducible, then the metric of V_n is equal to the metric of an anholonomic subspace of the space of G, endowed with its natural metric.

Since the space of a unitary group is a subspace of a sphere, Vrănceanu's conjecture is proved.

There follows a study of symmetric and closed riemannian spaces belonging to the class A I of Cartan [see Ann. Sci. École Norm. Sup. **44** (1927), 437–442].

R. Blum (Saskatoon, Sask.)

1632:

Blanuša, Danilo. A simple method of imbedding elliptic spaces in Euclidean spaces. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske Ser. II. **12** (1957), 189–200. (Serbo-Croatian summary)

The mapping $R^{n+1} \to R^{(n+1)^2}$ determined by $(x^i) \to \alpha(x^i x^j)$ $(i, j=1, \cdots, n+1; \alpha \neq 0)$ is two-to-one; points with opposite coordinates in R^{n+1} being sent into the same point of $R^{(n+1)^2}$. Considering the elliptic space El_n with curvature $1/\rho$ as obtained from the sphere S_n of radius ρ about the origin of R^{n+1}, and taking $\alpha = 1/\rho\sqrt{2}$, the restriction of the mapping to S_n provides an imbedding of El_n in $R^{(n+1)^2}$. The image set actually turns out to lie in R^N, $N=\frac{1}{2}n(n+3)$, and after introducing suitable coordinates in R^N this new, extremely elegant, imbedding is shown to be the same as the one studied in a few earlier papers [see, e.g., same Glasnik **10** (1955), 181–182; MR **18**, 146].

A. Nijenhuis (Seattle, Wash.)

1633:

Stavroulakis, Nicias. Points de rebroussement des surfaces et théorème de Gauss-Bonnet. C. R. Acad. Sci. Paris **245** (1957), 1112–1114.

A "point de rebroussement" is a singular point O on a 2-dimensionl surface with the property that all curves which lie on the surface and end at O have a common half-tangent OT at O. The Gauss-Bonnet theorem is extended to apply to regions which contain such singularities.

T. J. Willmore (Liverpool)

1634:

Mirguet, Jean. Nouvelles expressions intrinsèques de la dérivabilité seconde en théorie des surfaces. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 8–14.

Cette étude entre dans le cadre de la géométrie infinitésimale directe et s'inspire particulièrement de l'ouvrage de M. G. Bouligand [*Introduction à la géométrie infinitésimale directe*, Vuibert, Paris, 1932].

M. Bouligand a montré, dans le livre cité, qu'une surface S à plan tangent continu peut être représentée au voisinage d'un point M sous la forme $z=f(x, y)$ où la fonction f admet des dérivées partielles du second ordre continues, si sont satisfaites les hypothèses (a) et (b): (a) chaque demi-plan issu de la normale MZ contient un demi-cercle et un seul du contingent de courbure normale; (b) cette propriété subsiste en tout point d'un voisinage ouvert de M sur S.

M. Mirguet propose de remplacer la propriété (b) par une hypothèse (c) de "convergence moyenne et univoque des intersections": (c) si M_1' et M_2' sont deux points qui tendent simultanément sur S vers M, si A_2' est la projection orthogonale de M_2' sur le plan tangent en M_1' à S, le rapport de la distance du milieu de $M_1'A_2'$ à l'intersection des plans tangents en M_1' et M_2' à la distance $M_1'M_2'$ tend toujours vers zéro.

{La rédaction nous a paru manquer de clarté.}

M. Decuyper (Lille)

PROBABILITY

See also 1538, 1540, 1546, 1657, 1662, 1669, 1682, 1683a–b, 1723, 1724, 1844, 1919, 1920.

1635:

★**Kac, Mark. Probability and related topics in physical sciences.** With special lectures by G. E. Uhlenbeck, A. R. Hibbs, and B. van der Pol. Lectures in Applied Mathematics. Proceedings of the Summer Seminar, Boulder, Colo., 1957, Vol. I. Interscience Publishers, London-New York, 1959. xiii+266 pp. $5.60.

This volume consists of four chapters written by M. Kac and four appendices contributed by G. E. Uhlenbeck, A. R. Hibbs and Balth van der Pol. The first two appendices are due to Uhlenbeck and Hibbs respectively and complement Kac's discussion. Van der Pol has written the last two appendices and they are not directly related to the rest of the volume.

The first chapter is titled "Nature of Probabilistic Reasoning". After a brief discussion of elemental measure theoretics, Kac considers a variety of problems arising in a number of fields to illustrate the power of probabilistic reasoning. First the Maxwell distribution is derived. Then the average number of roots of an algebraic equation is computed, under the assumption that the coefficients are uniformly distributed on a sphere with the origin as center. A third problem is concerned with the computation of the "probability" distribution of the difference between the number of prime divisors of an integer n counting multiplicity and without multiplicity. Finally, a derivation of Borel's theorem on the average number of zeros in the binary expansion of a real number is given.

The second chapter, "Some Tools and Techniques of Probability Theory", is basically concerned with a set of random walk problems. The first random walk problem is a simplified model of a chain molecule with the links at random angles to each other. An ingenious reduction to a perturbation problem for an appropriately chosen integral equation yields an asymptotic result. The second problem concerns lattice random walks with independent increments. The close relationship between certain aspects of these random walks and corresponding problems on the asymptotic distribution of eigenvalues of Toeplitz forms is neatly sketched. This class of problems has recently attracted a good deal of attention.

The body of the book is concentrated in the third chapter, "Probability in Some Problems of Classical Statistical Mechanics". Aspects of the Maxwell-Boltzman approach to the kinetic theory of gases are discussed together with the objections of Zermelo. Much of the chapter is an attempt to clarify and reconcile these two. There follows a brief derivation of the Poincaré recurrence theorem. Several simple probabilistic models are discussed in detail in the hope of providing some insight into the apparent contradiction. A simplified model discussed in greatest detail is that of P. and T. Ehrenfest. A sketch of an analysis of R. Brout to justify the socalled "master equation" on the basis of Liouville's equation is given. The remainder of the chapter is devoted to an exposition of Smoluchowski's theory of fluctuations of concentration. The appendix of Uhlenbeck, "The Boltzman Equation", provides physical motivation and insight into the problems of this chapter and should perhaps be read beforehand.

The last chapter, "Integration in Function Spaces", considers the problem of obtaining the probability dis-

tribution of certain functionals of a Brownian motion process. The problem is reduced to that of finding the fundamental solution of a parabolic differential equation. This is compared to a parallel approach of R. Feynman in some problems of nonrelativistic quantum mechanics. The techniques are then applied to compute the partition function of a quantum mechanical system and obtain some results in classical potential theory. Hibbs' appendix on Quantum Mechanics sketches out the quantum mechanical background as it arises in such problems.

The two appendices of van der Pol deal with a smoothing problem and discrete potential theory.

The book for the most part retains the flavor of a set of lectures. Generally Kac's objective is that of avoiding the niceties of measure theory as they arise in probability theory and instead concentrating on the interesting analytic problems posed. The book is a stimulating addition to the literature on probability theory.

M. Rosenblatt (Providence, R.I.)

1636:

Kraft, Charles H.; Pratt, John W.; and Seidenberg, A. Intuitive probability on finite sets. Ann. Math. Statist. **30** (1959), 408–419.

Suppose that the subsets $\alpha, \beta, \cdots$ of a finite set, S, are ordered by a relation $\prec$, such that any pair of subsets can be compared ($\alpha \prec \beta$ or $\beta \prec \alpha$ or both), there is transitivity ($\alpha \prec \beta$ and $\beta \prec \gamma$ implies $\alpha \prec \gamma$), additivity (if γ is disjoint from both α and β, then $\alpha \prec \beta$ if and only if $\alpha \cup \gamma \prec \beta \cup \gamma$), and finally $\varphi \prec \gamma$ for every γ, where φ is the empty set. It was conjectured by de Finetti [Società Italiana per il Progresso delle Scienze, XLII Riunione, Roma, 1949, Relazioni, vol. I, pp. 227–236, Soc. Ital. Progr. Sci., Rome, 1951; MR **15**, 594], that every such ordering of subsets 'arises from a measure', i.e. that an additive measure can be ascribed to the subsets that will give the same ordering. The authors produce a counter-example to this conjecture, for a set, S, of five elements. They also modify the conjecture so as to make it true, while retaining, or even increasing, the intuitive appeal of the assumptions in their interpretation in terms of subjective probability.

I. J. Good (Middlesex)

1637:

Devinatz, A. On a theorem of Lévy-Raikov. Ann. Math. Statist. **30** (1959), 583–586.

Let φ_1 and φ_2 be two characteristic functions and suppose that $\varphi = \varphi_1\varphi_2$ is $2n$-times differentiable. The author shows that φ_1 and φ_2 are then also $2n$-times differentiable. Moreover there exist real numbers a_j, m_j, such that $|\varphi_j^{(2k)}(0)| \leqq m_j|\psi_{a_j}^{(2k)}(0)|$ for $\psi_a = e^{iax}\phi(x)$, $j=1, 2$ and $k = 1, 2, \cdots, n$. If φ is infinitely differentiable and if the Hamburger moment problem for the sequence $(-i)^k\varphi^{(k)}(0)$ is determined, then the same is true for the sequences $(-i)^k\varphi_j^{(k)}(0)$ for $j=1, 2$. The well-known theorem [due to D. A. Raikov, Izv. Akad. Nauk SSSR. Ser. Mat. **2** (1938), 91–124] which states that the analytic characteristic functions form a factor-closed family is obtained as a special case.

E. Lukacs (Washington, D.C.)

1638:

Lévy, Paul. Un paradoxe de la théorie des ensembles aléatoires. C. R. Acad. Sci. Paris **248** (1959), 181–184.

Let X be a random variable with values x in a space Ω and $y=f(x)$ be a function with values in a space Ω^*. The author shows that it is possible for the distribution of $Y=f(X)$ to be continuous (in the sense that any assigned single value of y of Y has probability 0) and yet to be such that there is no set $E^* \subset \Omega^*$ for which prob $(Y \in E^*)$ is both determinate and strictly between 0 and 1. He gives several examples, in which the Y's are random sets, to show that such cases can arise naturally and thus call for study. For instance, we may take for X a sequence $\{U_n\}$ of independent random real variables distributed continuously and identically, and for Y the set of the U_n's considered independently of their order.

H. P. Mulholland (Exeter)

1639:

Weiss, Mary. On the law of the iterated logarithm. J. Math. Mech. **8** (1959), 121–132.

Let $\chi_1, \chi_2, \cdots$ be a sequence of independent random variables with mean values zero and variances $b_1, b_2, \cdots$, respectively. Let

$$|\chi_n| \leqq M_n, \quad S_n = \sum_{\nu=1}^{n} \chi_\nu, \quad B_n{}^2 = \sum_{\nu=1}^{n} b_\nu \to \infty \ (n \to \infty).$$

According to the law of the iterated logarithm, if $M_n = o(H_n)$, where $H_n{}^2 = B_n{}^2/\log\log B_n$, then $L = \limsup S_n/(2B_n{}^2 \log\log B_n)^{1/2} = 1$ with probability 1. Refining a result obtained by J. Marcinkiewicz and A. Zygmund [Fund. Math. **29** (1937), 215–222], the author presents a theorem to the following effect: if (for $\nu \geqq 2$, say) χ_ν takes each of the values $\pm\exp\{\nu/(\log\nu)^2\}$ with probability $\alpha/\{2\log(\nu e^\alpha)\}$ and otherwise takes the value 0, and if α is sufficiently large, then $M_n/H_n \to K > 0$ and $L > 1$ with probability 1. {There are some inaccuracies in the details of the proof, but it may be possible to correct them. On p. 124 the terms in λ^4 and λ^6 involved in the first sum in (12) are incorrect, and in the last formula in the proof of Lemma 2 the remainder term is printed as $O(\lambda^2\omega)^2$, while the argument stated yields only $o(\lambda^2\omega)$ and we need $O(\lambda^2\omega)^3$. Consequently the calculations in pp. 123–129 need some revision.}

H. P. Mulholland (Exeter)

1640:

Sentis, Philippe. Les répartitions en classes et quelques-unes de leurs applications. Publ. Inst. Statist. Univ. Paris **7** (1958), 15–160.

Le présent travail développe principalement la notion d'ensemble classifié. Au premier chapitre l'auteur montre que toutes les variables aléatoires dont les valeurs sont fixées par un même évènement peuvent s'exprimer comme des fonctions non aléatoires d'un ensemble de variables dont on peut fixer arbitrairement les valeurs; si en outre les variables initiales se prêtent à une étude statistique, on peut les exprimer comme des fonctions non aléatoires de variables aléatoires indépendantes.

Le deuxième chapitre précise ce qu'on entend par variables qui se prêtent à une étude statistique; ce sont celles dont on peut répartir les valeurs en classes. Si l'on sait définir plusieurs répartitions en classes sur l'ensemble des valeurs de la variable, on dit que cet ensemble possède une structure d'ensemble classifié.

Les ensembles classifiés possèdent des propriétés qui font l'objet des chapitres 3 et 4. Ces propriétés permettent de définir des fonctionnelles σ-additives et π-multiplicatives par une généralisation des indices par des intégrales et des

mesures. L'auteur réussit ensuite à introduire une nouvelle espèce de multiplication d'une variable aléatoire par un nombre qui permet de définir l'intégral d'un fonction à valeurs aléatoires indépendantes et surtout la dérivé d'un processus de Wiener-Lévy.

Les chapitres 5 et 6 sont consacrés à des applications.

A. Fuchs (Strasbourg)

1641:

Ambarcumyan, G. A. Entropy of Markov chains. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 2, 31–40. (Russian. Armenian summary)

The equi-partition principle for multiple Markov chains is given. This result is only apparently more general than that for simple Markov chains by the well-known reduction from the multiple to simple case.

K. L. Chung (Syracuse, N.Y.)

1642:

Vere-Jones, D.; and Kendall, David G. A commutativity problem in the theory of Markov chains. Teor. Veroyatnost. i Primenen. **4** (1959), 97–100. (Russian summary)

Let P be the matrix of one-step transition probabilities for a Markov chain with finitely many states and let Π be the $(C, 1)$ limit of P^n. The matrices P and Π can be identified with bounded linear operators acting on the Banach space L of absolutely convergent series. Let A denote an arbitrary bounded linear operator in L that commutes with P. Then the commutability of operators Π and A holds if and only if either (a) $\Pi=0$, or (b) $\Pi=\lim_{n\to\infty}(P+P^2+\cdots+P^n)/n$ in the sense of the norm. The property of commutability always is satisfied if the set of states of the Markov chain is finite and also $\Pi=\phi(P)$ where ϕ is a certain polynomial. The first theorem gives the rule for finding this polynomial.

H. P. Edmundson (Pacific Palisades, Calif.)

1643:

Ray, Daniel. Stationary Markov processes with continuous paths. Trans. Amer. Math. Soc. **82** (1956), 452–493.

This paper examines relations between regularity properties of transition probabilities $P(x, t, E)$ on the real line, regularity properties of the semigroup of operators and the resolvent gotten from these, realizability of the process on a path space with no jumps except to $\pm\infty$, and a strong Markov property. First, some possible properties: (A_0) if $T_tf(x)=\int f(y)P(x, t, dy)$, then f bounded, continuous $\Rightarrow T_tf$ continuous;

$$(B_0)\qquad \lim_{t\to 0} t^{-1}P(x, t,]x-\delta, x+\delta[') = 0,$$

all $\delta>0$; (C_0) the limit in (B_0) is uniform in x if x remains bounded. Let Ω_x be those functions ω from $[0, +\infty[$ to $[-\infty, +\infty]$ with $\omega(0)=x$ and such that, given $\varepsilon>0$, there exists a $\delta>0$ for which $|\omega(t)|<1/\varepsilon$, $|t-t'|<\delta\Rightarrow|\omega(t)-\omega(t')|<\varepsilon$. By ($\alpha$) we mean that there exists, for each x, a probability measure $\Pr\{\cdot|x\}$ on Ω_x such that $x_t(\omega)=\omega(t)$ is a Markov process with transition probabilities P starting at x. Given such a realization, let $P_0(t, x;]a_1, a_2[, a_j)$ be the probability that the process beginning at x has first left $]a_1, a_2[$ by time t, via exit a_j. The point $y>x$ is called accessible from the left if there exists a finite $a<x$ such that $P_0(t, x;]a, y[, y)>0$ (and analogous definition from right). The first passage time relation (π) for a process satisfying (α) is that for each bounded Borel function f we have

$$E\{f(x_t)|x\} = \sum_{j=1,2}\int_0^t E\{f(x_{t-t'})|a_j\}P_0(dt', x;]a_1, a_2[, a_j)$$

for all $t>0$, $-\infty<a_1<x<a_2<+\infty$.

Theorem I. Let (α) hold, and $R_sf(x)=\int_0^\infty E\{f(x_t)|x\}e^{-st}dt$. (A) Suppose for all $s>0$ and all y which are left [right] accessible with respect to some $\Pr\{\cdot|x\}$, R_sf is right [left] continuous at y. Then (π) holds. (B) Conversely, if (π) holds, then for all $s>0$ and all bounded Borel measurable f for which $f(x+)$ [$f(x-)$] exists, R_sf has a right [left] limit at x. Likewise for limits at $\pm\infty$. R_sf is continuous whenever f is, except for countably many exceptional points independent of f and s, all both right and left inaccessible for all starting points x.

Cor. I.1: (α), (π)$\Rightarrow$(C_0). Cor. I.2: (A_0), (α)$\Rightarrow$(π), and hence also (C_0). Cor. I.3 connects the behavior of f at $\pm\infty$ with that of T_tf and R_sf.

Theorem II: (α) holds if either (A_0) and (B_0), or (C_0) hold. It had been conjectured by Feller that, given (A_0), then (α) and (B_0) are equivalent. This fact is contained in the above results. Various examples are given to disprove some other possible conjectures about relationships of the properties discussed.

J. Feldman (Berkeley, Calif.)

1644:

Bass, J. Contribution à l'étude de certaines fonctions susceptibles de représenter la vitesse d'un fluide turbulent. J. Math. Pures Appl. (9) **37** (1958), 173–205.

The paper starts with a lucid exposition of the difficulties encountered in the theory of turbulence if one tries to give a mathematical meaning to the rather vague notion of "complication" for a function $u(t)$; the author brings specially into focus the difficulties concerning harmonic analysis. His aim is to define a class of functions which are "complicated" and nevertheless able to be subjected to harmonic analysis in some suitable way. Having in mind the function considered by N. Wiener [*The Fourier integral*, University Press, Cambridge, 1933; Dover, New York, 1959; MR **20** #6634; see p. 151] the author introduces the class of step-functions

$$\text{(a)}\qquad u(t) = 0 \quad (t<0),$$

$$\text{(b)}\qquad u(t) = \exp \pi i\varphi(n) \quad (n \leq t < n+1),$$

where $\varphi(t)=\alpha_qt^q+\cdots+\alpha_0$, α_q/π irrational.

It is proved that:

$$\text{(1)}\qquad \bar{u}(t) = \lim T^{-1}\int_0^T u(t)dt = 0;$$

$$\text{(2)}\qquad \gamma(h) = \lim T^{-1}\int_0^T u(t)u(t+h)dt \quad (T\to\infty)$$

exists for all h; (3) $\gamma(h)$ is continuous for all h; (4) $\gamma(0)>0$ and (5) $\gamma(\infty)=0$. Thus the generalized harmonic analysis of N. Wiener can be applied to $u(t)$.

When a function $u(t)$ has the properties (1), (2), (3), (4), (5), it is said that it belongs to the class A; $g(\lambda)$ being of bounded total variation on $[-\infty, +\infty]$, the author proves that if $u(t)\in A$ and is bounded, then

$$f(t) = \int_{-\infty}^{+\infty} u(t+\lambda)dg(\lambda) \quad \in A.$$

In this way, from a step-function $u(t)\in A$, one can deduce functions $f(t)\in A$ which are no longer step-functions; for the applications that the author has in view a special interest is attached to the case where $f(t)$ is continuous and continuously differentiable; the following result is typical in this way: $u(t)$ being defined by (a) and (b), if the polynomial $\varphi(t)$ is an increasing function of t and if $dg(\lambda)=s(\lambda)d\lambda$, then $f(t)$ is continuous.
J. Kampé de Fériet (Lille)

1645:

Bass, Jean; et Krée, Paul. Sur les fonctions pseudo-aléatoires. C. R. Acad. Sci. Paris **247** (1958), 1083–1085.

The authors suggest calling "fonction pseudo-aléatoire" the functions belonging to the class which Bass has previously introduced [Bass, same C. R. **245** (1957), 1217–1219; MR **20** #2828; Bass and Bertrandias, ibid., 2457–2459; MR **20** #2829], and give some new rules for defining functions of this type. The main result is: Given two polynomials $\varphi(t)=At^q+\cdots$, $\psi(t)=Bt^q+\cdots$, $q\geq 2$, if there are no integers k_1, k_2, k_3, k_4 such that $k_1A+k_2\alpha+k_3B=k_4$, then

$$f(t) = \exp 2\pi i[\varphi(\hat{t})+\psi(\hat{s})]$$

is pseudo-aléatoire, where $s=\alpha t+\beta$ and $\hat{\kappa}$ is the greatest integer not greater than κ. *J. Kampé de Fériet* (Lille)

1646:

Bass, Jean. Fonctions pseudo-aléatoires et fonctions de Wiener. C. R. Acad. Sci. Paris **247** (1958), 1163–1165.

For all the fonctions pseudo-aléatoires constructed previously by the author [see review above] the values $f(t)$ are dense on the circle $|f(t)|=1$. A method is described here leading to functions taking only a finite number of values on the same circle. *J. Kampé de Fériet* (Lille)

1647:

Bass, J. Sur la définition temporelle des fonctions aléatoires. Publ. Inst. Statist. Univ. Paris **6** (1957), 199–211.

L'auteur se pose la question de savoir de quelle façon une seule épreuve sur une fonction aléatoire stationnaire permet de définir cette fonction, plus précisément, de définir sa loi temporelle. Soit $u(t)$, $t\geq 0$ une réalisation d'une fonction aléatoire stationnaire $U(t)$. Conformément aux idées que sont à la base des théorèmes ergodiques, l'auteur substitue aux valeurs $E\{U^n(t)\}$, $E\{U(t)U(t+h)\}$, $\cdots$ les moyennes expérimentales

$$T^{-1}\int_0^T u^n(t)dt,\quad T^{-1}\int_0^T u(t)u(t+h)dt$$

portant sur un intervalle de temps T assez grand. Il est alors amené à introduire la fonction

$$\varphi(z_1, z_2) = T^{-1}\int_0^T \exp\{i[z_1u(t)+z_2v(t)]\}dt,$$

à examiner dans quelles cas c'est une fonction caractéristique d'un couple aléatoire, et à en étudier les propriétés. *A. Fuchs* (Strasbourg)

1648:

Bertrandias, Jean-Paul. Formation d'une classe de fonctions pseudo-aléatoires. C. R. Acad. Sci. Paris **248** (1959), 513–515.

The author considers a new class of the functions "pseudo-aléatoires" previously introduced by J. Bass, P. Krée and himself [see the three preceding reviews] based on the following theorem. If

$$\text{(a)}\qquad \psi(t) = At^\nu+Bt^{\nu-1}+\cdots,$$

$\nu\geq 2$, A irrational, (b) $\varphi(n)$ a function of n (integer) such that the points having as $p+2$ coordinates $\varphi(n+q)-\varphi(n), \psi(n), \psi(n+1), \cdots, \psi(n+p)$ are uniformly distributed modulo 1 in the cube C_{p+2} for every q and $p\leq\nu-1$, (c) $r(z)\geq 0$, $0\leq z\leq 1$, bounded and Riemann-integrable; then the function

$$f(t) = \exp[2\pi i\varphi(n)]\quad\text{in}\quad t_{n-1}\leq t<t_n,$$

(where $t_n-t_{n-1}=r(x_n)$, $x_n=\psi(n)-m$, m greatest integer $\leq\psi(n)$) is pseudo-aléatoire. *J. Kampé de Fériet* (Lille)

1649:

Takács, L. On a combined waiting time and loss problem concerning telephone traffic. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **1** (1958), 73–82.

The queueing system $GI/M/m$ is studied under the condition that the total capacity of the system cannot exceed $m+w$. Any calls received when the system is full are lost. Using the usual imbedded Markov chain argument the limiting distribution of the number of customers in the system at arrival instants is found. Blackwell's renewal theorem is applied to find the limiting distribution of the number of customers in the system, provided the interarrival distribution is not of lattice type. Finally, the conditional distribution function of the waiting time of an arbitrary call that is not lost is determined.
D. V. Lindley (Cambridge, England)

1650:

Conolly, B. W. The busy period in relation to the queueing process GI/M/1. Biometrika **46** (1959), no. 1/2, 246–251.

A single server queueing process is considered in which the inter-arrival times and the service times are independent sequences of identically distributed independent random variables with density functions $a(t)$ and $e^{-t/b}b$ $(0\leq t<\infty)$, respectively. Denote by $P_m(t)$ the probability that a busy period consists of m services and its length is $\leq t$. Write

$$\pi_m(s) = \int_0^\infty e^{-st}dP_m(t)\quad\text{and}\quad \alpha(s) = \int_0^\infty e^{-st}a(t)dt$$

if $\mathrm{Re}(s)\geq 0$. The author shows that if $\mathrm{Re}(s)>0$ and $|y|\leq 1$ then

$$\sum_{m=1}^\infty \pi_m(s)y^m = \frac{y(1-\xi)}{bs+1-y\xi},$$

where $\xi=\xi(s,y)$ is the only root of the equation $\xi=\alpha(s+(1-y\xi)/b$ for which $|\xi|<1$. This result was proved earlier by F. Pollaczek [*Problèmes stochastiques posés par le phénomène de formation d'une queue d'attente à un guichet et par des phénomènes apparentés*, Gauthier-Villars, Paris, 1957; MR **19**, 987; p. 102]. *L. Takács* (London)

1651:

Takács, L. On the limiting distribution of the number of coincidences concerning telephone traffic. Ann. Math. Statist. **30** (1959), 134–142.

A telephone exchange with an infinite number of available channels is defined to be in state E_k if exactly k channels are busy. Interarrival periods of calls are independent, identically distributed random variables with distribution $F(x)$, possessing finite mean and variance. Define $\varphi(s)=\int_0^\infty e^{-sx}dF(x)$. Durations of calls are assumed to be mutually independent random variables, each with the same exponential distribution $1-e^{-\mu x}, x\geq 0$. The author's problem is to compute the distribution of $\nu_t^{(k)}$, the number of transitions $E_k\rightarrow E_{k+1}$ occurring in the time interval $(0, t]$, especially as $t\rightarrow\infty$. It turns out that $\Pr\{\nu_t^{(k)}>n\}=R_k^* *$ (n-fold convolution of R_k), where $R_k(t)$, $R_k^*(t)$ are certain auxiliary distributions whose Laplace transforms the author succeeds in expressing in terms of $\varphi(s)$. The asymptotic distribution of $\nu_t^{(k)}$, as $t\rightarrow\infty$, is normal, with a mean and variance that can be expressed in terms of $\varphi(s)$.

E. Reich (Minneapolis, Minn.)

1652:

Gupta, H. C. Diffusion by continuous movements. J. Math. Phys. **38** (1959/60), 36–41.

Consider a particle moving on the line subject to the following rule: there are n permissible speeds; starting at a point $x(0)$ the particle moves at a permissible speed $\dot{x}(t)=\dot{x}(0)$ up to a time t_1 with distribution $P(t_1>t|\dot{x}(0))=e(-\text{constant } t)$ in which the constant depends upon $\dot{x}(0)$; at time t_1, $\dot{x}(t)$ jumps from $\dot{x}(0)$ to another permissible speed $\dot{x}(t_1)$ and the particle starts from scratch, moving at speed $\dot{x}(t)=\dot{x}(t_1)$ up to time t_1+t_2; etc. The process thus defined is not Markovian, but we can make it so if we think of it as taking place on n copies of the line, one to each permissible speed, and think of the jumps in speed as jumps from one line to another.

The author considers motions of this kind subject to the creation and destruction of mass, derives the forward equation for the transition probabilities, and integrates it in the case $n=2$.

S. Goldstein [Quart. J. Mech. Appl. Math. **4** (1951), 129–156; MR **13**, 960] treated the special case $n=2$ with permissible speeds $\pm v$ and conservation of mass which leads to the telegraphist's equation with no leakage.

H. P. McKean, Jr. (Cambridge, Mass.)

STATISTICS

See also 1916, 1919.

1653:

★Zimmermann, K. F. Tabellen, Formeln und Fachausdrücke zur Variationsstatistik: Für Landwirtschaftswissenschaftler, Naturwissenschaftler, Mediziner und Ingenieure. VEB Deutscher Verlag der Wissenschaften, Berlin, 1959. 129 pp. DM 12.80.

This book contains tables of the usual statistical distribution functions and also special tables applicable to agricultural experiments. There is a short dictionary of statistical terms in German, English and Russian.

G. Tintner (Ames, Iowa)

1654:

Mauldon, J. G. A generalization of the beta-distribution. Ann. Math. Statist. **30** (1959), 509–520.

The author defines and studies a class of distributions which includes as particular cases the ordinary β-distribution, the (univariate) triangular distribution, the uniform distribution over any nondegenerate simplex, and a continuous range of other distributions over such a simplex. His class also includes various (univariate and other) distributions that pertain to the random division of an interval. An application is the test of the hypothesis H_0 that $n-1$ numbers $y_1, \cdots, y_{n-1}$ all in the unit interval were drawn independently from a rectangular distribution over (0, 1). Moreover, the power functions of certain tests of the hypothesis H_0 can be found.

H. P. Edmundson (Pacific Palisades, Calif.)

1655:

Glasgow, M. O. Note on the factorial moments of the distribution of locally maximal elements in a random sample. Ann. Math. Statist. **30** (1959), 586–590.

Dans un ensemble ordonné, E, de n nombres réels inégaux, un élément, x, est dit localement k-maximal s'il existe un sous-ensemble de k éléments consécutifs $F\subseteq E$, tel que x soit le plus grand des éléments de F. Une méthode de dénombrement des permutations de n éléments d'après le nombre des divers éléments localement maximaux avait été donnée par T. Austin, R. Fagen, T. Lehrer et W. Penney [same Ann. **28** (1957), 786–790; MR **19**, 936]. Ces résultats sont étendus ici et conduisent à une relation de récurrence générale pour les moments factoriels de la distribution. La méthode appliquée est encore utilisable pour tout autre moment factoriel d'un ordre plus élevé. Les résultats peuvent être vérifiés au moyen des exemples donnés par Austin et al. (loc. cit.).

A. Sade (Marseille)

1656:

Zoroa, P. Convolution of histograms. Trabajos Estadist. **9** (1958), 159–182. (Spanish. English summary)

An histogram can be considered as a special function of a continuous random variable.

It is possible by means of certain auxiliary functions to give formulas for the convolution of distributions defined by histograms.

These formulas are generalized for histograms in n dimensions. Two numerical examples are given.

Author's summary

1657:

Jones, Howard L. How many of a group of random numbers will be usable in selecting a particular sample? J. Amer. Statist. Assoc. **54** (1959), 102–122.

The number of usable random numbers, s, is equivalent to the number of occupied cells in a subclass of T cells of a class of N cells among which n objects are distributed at random. The distribution of s is obtained, and some approximations to this distribution are suggested and investigated.

D. M. Sandelius (Göteborg)

1658:

van Eeden, Constance. Note on two methods for estimating ordered parameters of probability distributions. Nederl. Akad. Wetensch. Proc. Ser. A **60** = Indag. Math. **19** (1957), 506–512.

A proof is given that the maximum likelihood estimations of ordered probabilities described in [A] and [B] are equivalent. [A]: van Eeden, Math. Centrum Amsterdam,

Statist. Afdeling. Rep. **S 188** (VP5) (1956); Rep **S 196** (VP7) (1956) [MR **17**, 640, 982]. The same material appeared in Nederl. Akad. Wetensch. Proc. Ser. A, **59** = Indag. Math. **18** (1956), 444–455 [MR **18**, 772]. [B]: Ayer, Brunk, Ewing, Reid and Silverman, Ann. Math. Statist. **26** (1955), 641–647 [MR **17**, 504].

M. Dwass (Evanston, Ill.)

1659:

van Eeden, Constance. A least squares inequality for maximum likelihood estimates of ordered parameters. Nederl. Akad. Wetensch. Proc. Ser. A **60** = Indag. Math. **19** (1957), 513–521.

The inequality of line 20, p. 644 of reference [B] above is generalized. This is a technical device for maximizing likelihood functions, over parameter sets satisfying certain order constraints. *M. Dwass* (Evanston, Ill.)

1660:

★van Eeden, Constance. Testing and estimating ordered parameters of probability distributions. Mathematical Centre, Amsterdam, 1958. 124 pp. $2.50.

Let $X_1, \cdots, X_k$ be independent random variables whose distributions depend on parameters $\theta_1, \cdots, \theta_k$, respectively. Let $\varphi_1(\theta_1), \cdots, \varphi_k(\theta_k)$ be real-valued functions, and suppose it is known a priori that (a) $\varphi_1(\theta_1), \cdots, \varphi_k(\theta_k)$ are partially or completely ordered, and (b) $\varphi_i(\theta_i)$ lies in I_i, a specified interval ($i=1, \cdots, k$). In the case of $\varphi_i(\theta)=\theta$, I_i the unit interval, complete ordering, $\theta_1 \leqq \cdots \leqq \theta_k$, and X_i binomially distributed with probability parameter θ_i, maximum likelihood estimates for the θ_i were described in [B] (reference described two reviews above). This monograph now characterizes maximum likelihood estimates in the more general case. Computational techniques are described and examples explicitly worked for $\varphi_i(\theta) = \alpha_i + \beta_i\theta$, and the X_i's binomial, normal, Poisson, exponential and rectangular. The main tool in describing these estimates is a characterization of the maximum of a function over a convex set. There is a proof of consistency and a description of a class of tests of the hypothesis of complete ordering, $\theta_1 \leqq \cdots \leqq \theta_k$.

M. Dwass (Evanston, Ill.)

1661:

Madansky, Albert. The fitting of straight lines when both variables are subject to error. J. Amer. Statist. Assoc. **54** (1959), 173–205.

Expository paper. *D. M. Sandelius* (Göteborg)

1662:

Harris, Bernard. Determining bounds on integrals with applications to cataloging problems. Ann. Math. Statist. **30** (1959), 521–548.

In a random sample of size N from a multinomial distribution of unknown index (possibly infinite) and parameters $\{p_j\}$ let n_r denote the number of classes which occur exactly r times, $d=\sum n_r$ and $C=\sum p_j$ where the final summation is over those classes for which at least one representative has been observed. The means and variances of d, n_r and C are found and exponential approximations to the means obtained. Let $d(\alpha)$ and $C(\alpha)$ be the values of d and C obtained, either in a second sample of size αN, or in an augmented sample of size $(\alpha-1)N$, $\alpha \geqq 1$. The expectations of $d(\alpha)$ and $C(\alpha)$ are expressed in the forms

$$Ed(\alpha) \sim d + n_1 E\left\{\frac{1-e^{-(\alpha-1)x}}{x}\right\},$$

(*)

$$EC(\alpha) \sim 1-\frac{n_1}{N}+\frac{n_1}{N}E\{1-e^{-(\alpha-1)x}\},$$

where the expectations on the right are with respect to an unknown distribution on $[0, \infty)$ whose sample moments are known. The author suggests using bounds to these expectations, and hence to $Ed(\alpha)$ and $EC(\alpha)$, based on the class of distributions having moments equal to the sample ones. Most of the paper is devoted to a solution of this problem, for a general function $g(x)$, and later particularized to the special $g(x)$ on the right of (*). It is shown that with the first k moments specified the extremals can be found by solving a system of k equations. These are discussed in detail for $k \leqq 3$. These results are then applied to (*) and compared with the work of Corbet, Fisher and Williams [J. Animal Ecology **12** (1943), 42–58], Goodman [Ann. Math. Statist. **20** (1949), 572–579; MR **11**, 260] and Good and Toulmin [Biometrika **43** (1956), 45–63; MR **17**, 982]. Three numerical examples conclude the paper.

D. V. Lindley (Cambridge, England)

1663:

★Kiveliovitch, M. L'estimation de la part du hasard. Actes des colloques de calcul numérique, Caen, 1955; Dijon, 1956, pp. 1–8. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 77, Paris, 1958. vi+144 pp. 2105 francs.

Summary of joint work of the author and J. Villar [J. Sci. Météorol. **5** (1953), 21–46, 75–87, 89–101, 129–143; **6** (1954), 1–16, 73–83, 151–166; **7** (1955), 259–271; MR **16**, 941, 942; **17**, 1102, 1221]. *A. Dvoretzky* (Jerusalem)

1664:

Aggarwal, Om P.; and Guttman, Irwin. Truncation and tests of hypotheses. Ann. Math. Statist. **30** (1959), 230–238.

X is said to have a "symmetrically truncated" normal distribution if there is $a>0$ for which the density of X is zero outside the interval $\{x \mid |x-\mu| < a\sigma\}$ and is a constant multiple of the $N(\mu, \sigma^2)$ density interior to the given interval. The authors consider the effects of symmetric truncation upon the size and power of the UMP test of the hypothesis $\mu=\mu_0$ against $\mu>\mu_0$ for non-truncated normal distribution with known σ^2. Tables are presented giving loss in power and changes in size resulting from various symmetric truncations. In addition a UMP test recognizing the symmetric truncation is produced and a table giving the gain in power of this test over that of the original test is also presented. The authors in general conclude that symmetric truncation introduces serious losses in size rather than in power.

F. C. Andrews (Eugene, Ore.)

1665:

Weiss, Lionel. The limiting joint distribution of the largest and smallest sample spacings. Ann. Math. Statist. **30** (1959), 590–593.

Let $X_1, X_2, \cdots, X_n$, be independently and identically distributed over the interval (0, 1) with common density $f(x)$. If U_n is the smallest and V_n the largest of the

subintervals into which the unit interval is divided by the X_i, the author calculates the limiting joint distribution of U_n and V_n for $f(x)$ a piecewise constant positive density. This result is used to calculate the asymptotic power of tests that $f(x)=1$, $0<x<1$ against alternatives $f(x)$ piecewise constant, using tests based on the values of V_n/U_n and V_n-U_n.
D. A. Darling (Santa Monica, Calif.)

1666:

Mitra, Sujit Kumar. On the limiting power function of the frequency chi-square test. Ann. Math. Statist. **29** (1958), 1221–1233.

A χ^2 statistic is considered for testing the hypothesis that in q independent, multinomially distributed random vectors the "cell" probabilities are given functions of several unknown parameters, asymptotically efficient estimators of which are used in the statistic. By a method similar to Cramér's [*Mathematical methods of statistics*, Princeton Univ. Press, Princeton, N.J., 1946; MR **8**, 39], it is shown that if the true cell probabilities differ from the hypothetical ones by additive terms which suitably decrease to zero as the sample size increases, and regularity conditions analogous to Cramér's are satisfied, then the statistic has a limiting non-central chi-square distribution. The result is applied to two problems in the planning of experiments.
W. Hoeffding (Chapel Hill, N.C.)

1667:

Lawley, D. N. Tests of significance in canonical analysis. Biometrika **46** (1959), no. 1/2, 59–66.

Let X_1, X_2 denote random vectors of dimensions p and q respectively, where it is assumed that all random variables follow multivariate normal distributions with zero means, and with variances and covariances of the following form:

$$E(X_1X_1')=I_p,\quad E(X_2X_2')=I_q,$$
$$E(X_1X_2')=P,\quad E(X_2X_1')=P',$$

where I_p denotes the unit matrix of order p, P is a $p\times q$ matrix with diagonal elements equal to ρ_i and non-diagonal elements zero. Given that k of these canonical correlation coefficients ρ_i are distinct and positive, the problem considered is that of testing that the remainder are zero. Bartlett [Proc. Cambridge Philos. Soc. **34** (1938), 33–40] has given an approximate test based on the statistic $-\log\sum_{i=k+1}^{q}(1-r_i^2)$ where the r_i are the sample canonical correlation coefficients corresponding to the ρ_i. This statistic when multiplied by a suitable factor has an approximate χ^2 distribution. By obtaining expectations of various statistics by expansions in series to terms of order n^{-2}, a slight improvement is obtained in the multiplying factor suggested by Bartlett. As the author notes this involves "rather tedious and laborious algebra" and he further adds "no attempt has been made to justify the various expansions in series; they are presumed to be valid in an asymptotic sense."
D. G. Chapman (Seattle, Wash.)

1668:

Kiefer, J. K-sample analogues of the Kolmogorov-Smirnov and Cramér-V. Mises tests. Ann. Math. Statist. **30** (1959), 420–447.

Let $X_{j1}, X_{j2}, \cdots, X_{jn_j}$ be random variables having a common unknown continuous distribution function $F_j(x)$ and empirical distribution function $S_j(x)$, $j=1, 2, \cdots, k$, the $n_1+\cdots+n_k$ random variables being mutually independent. The author considers the general homogeneity hypothesis $\mathrm{H}_1: F_1=F_2=\cdots=F_k$, and the goodness-of-fit hypothesis $\mathrm{H}_2: F_1=F_2=\cdots=F_k=G$ where G is specified, introducing 5 test statistics which are natural analogues of the tests of the title. For the test of H_1 the following two ways are studied:

$$T=\sup_x\sum_j C_j(S_j(x)-\bar{S}(x))^2,$$

$$W=\int\sum_j C_j(S_j(x)-\bar{S}(x))^2d\bar{S}(x),$$

where $\bar{S}(x)$ is the empirical distribution function of the k pooled samples and the C_j are certain normalizing constants. The two corresponding test statistics for H_2 are T', W', respectively, obtained by replacing $G(x)$ for $\bar{S}(x)$ in T, W.

The limiting distributions of T, W, T', W' if H_1 and H_2 are true are the same as the distributions of the corresponding functionals of a normalized multi-dimensional Wiener-Lévy process which, modifying slightly the approach of Kac for the case $k=1$, can be found in terms of the Green's solution of certain classical differential equations. These solutions can be expressed in terms of known transcendents, i.e., Bessel functions, Weber functions, etc., which form the computational basis for a set of tables of the limiting distribution at the end of the paper. There are a few additional remarks on the power of the tests, possible extensions, and related criteria whose distributions can be obtained from known results.
D. A. Darling (Santa Monica, Calif.)

1669:

Kemperman, J. H. B. Asymptotic expansions for the Smirnov test and for the range of cumulative sums. Ann. Math. Statist. **30** (1959), 448–462.

Consider a random walk $\{z_n: n\geqq 0\}$ whose independent increments take on the values ± 1, each with probability $\frac{1}{2}$. Set $p_n(i, j, c)=\Pr[z_n=j,\ 0<z_m<c,\ m=1, \cdots, n|z_0=i]$, where i, j, c, n are positive integers. Using the known exact expression for $p_n(i, j, c)$, a short proof of which is included, the author develops an asymptotic expansion for it which yields, for each integer m, an approximation to $p_n(i, j, c)$ with absolute error less than Cn^{-m} where C does not depend upon i, j, c, n. This approximation is of the form

$$(2/\pi)^{1/2}\sum_{k=0}^{m-1}n^{-k-1/2}\sum_{h=0}^{k}(-1)^hA_{kh}g_{k+h},$$

where A_{kh} is defined in terms of the first k Bernoulli numbers and where g_r is an infinite series whose summands are essentially values of the Hermitian polynomial of degree $2r$. This result is applied to the random walk which arises in Smirnov's two-sample test (equal sample sizes) extending the earlier work of Gnedenko [Dokl. Akad. Nauk SSSR **82** (1952), 661–663; MR **13**, 760]. The author also derives both exact and asymptotic expressions for the distribution function of the range of the random walk's first n values $z_1, z_2, \cdots, z_n$.
R. Pyke (New York, N.Y.)

1670:

★Линник, Ю. В. **Метод наименьших квадратов и основы математико-статистической теории обработки наблюдений.** [Linnik, Yu. V. **The method of least squares and the foundations of the matematico-statistical theory of reduction of observations.**] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 333 pp. 12.15 rubles.

An elegant exposition from first principles, starting with an introductory chapter on matrices, which are used extensively. In chapters II and III are set forth the "necessary" principles from probability and from statistics. Thereafter the subject itself is developed by easy stages, and with many examples, but without letting the examples carry the main burden of the exposition. Hence although the examples come largely from geodesy, they need not repel anyone whose field of application lies elsewhere. *A. S. Householder* (Oak Ridge, Tenn.)

1671:

Sprott, D. A. A series of symmetrical group divisible incomplete block designs. Ann. Math. Statist. **30** (1959), 249–251.

Using the fact that $PG(2, s^2)$ contains $PG(2, s)$ where s is a prime power, the author shows how to construct a group divisible design [Bose and Connor, same Ann. **23** (1952), 367–383; MR **14**, 124] with parameters $v=b=s^4-s$, $r=k=s^2$, $\lambda_1=0$, $\lambda_2=1$, $m=s^2+s+1$, $n=s^2-s$. *S. S. Shrikhande* (Chapel Hill, N.C.)

1672:

Shah, B. V. A note on orthogonality in experimental designs. Calcutta Statist. Assoc. Bull. 8 (1958), 73–80.

A design for obtaining information on several sets of parameters is said to be an orthogonal design if the estimates of estimable parameters of the different groups are uncorrelated. The author derives necessary and sufficient conditions for the orthogonality of a design in which inhomogeneity is eliminated in one or in two directions. A suitable measure for non-orthogonality is defined and explicitly computed for balanced and also for partially balanced incomplete block designs.

H. B. Mann (Columbus, Ohio)

1673:

Shrikhande, S. S. On a characterization of the triangular association scheme. Ann. Math. Statist. **30** (1959), 39–47.

For definitions see W. S. Connor [same Ann. **29** (1958), 262–266; MR **20** #3620]. The author proves the following theorem: "A necessary and sufficient condition that a partially balanced incomplete block design with two associate classes for $n(n-1)/2$ treatments with $p_{11}{}^1=n-2$ has a triangular association scheme, is that the first associates of every treatment can be divided into two sets $(y_1, \cdots, y_{n-2})$ and $(z_1, \cdots, z_{n-2})$ such that y_i and y_j, $i\neq j$, are first associates and z_i and z_j, $i\neq j$, are first associates." This theorem is equivalent to a theorem of Connor (loc. cit.), but the author proves it independently and uses it to establish the triangularity of certain partially balanced incomplete block designs.

H. B. Mann (Columbus, Ohio)

1674:

Good, I. J. The interaction algorithm and practical Fourier analysis. J. Roy. Statist. Soc. Ser. B. **20** (1958), 361–372.

The algorithms introduced by Yates (1937) for calculating the interactions of 2^n factorial experiments starting with the original observations and then extended by Box et al. (1954) to the 3^n factorial experiments are treated from the viewpoints of matrix theory. The first theorem on matrix theory describes the direct algorithm useful for t^n factorial experiments, while the second theorem describes the inverse algorithm.

The method of obtaining the divisors for the analysis of variance is also discussed. A later paragraph is concerned with the connection with practical Fourier analysis. Multidimensional transforms with all moduli equal, general multidimensional transform, one-dimensional transform and multidimensional transform with composite moduli are discussed in the Fourier transform.

T. Kitagawa (Fukuoka)

1675:

Isbell, J. R. On a problem of Robbins. Ann. Math. Statist. **30** (1959), 606–610.

Given two coins with unknown probabilities p_1, p_2 of coming up heads, the problem is to describe a rule for choosing at each step one of the coins to toss as a function of the results of the last r tosses, so as to maximize the frequency of heads. The author regards the sequence of tosses as a Markov process with 4^r states (of the memory). Given a rule R and an initial state i, the frequency of heads converges to a limit $f(R, i, p_1, p_2)$. The worth of R is defined as the smaller of $\min_i f(R, i, p_1, p_2)$ and $\min_i f(R, i, p_2, p_1)$. Let $R_r{}^*$ be the rule which prescribes to change coins when one coin shows tails r successive times, or when $r-1$ tails with one coin are followed by a tail with the other coin. It is shown that the worth of $R_r{}^*$ is never smaller than the worth of the rule with the same r considered by Robbins [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 920–923; MR **18**, 606] and is not smaller than the worth of any other rule with the same r in some special cases. *W. Hoeffding* (Chapel Hill, N.C.)

1676:

Klega, Vladimír. Statistical quality control of out-of-roundness of machined parts. Apl. Mat. **4** (1959), 109–125. (Czech and Russian summaries)

It is assumed that a quantity X used to measure the 'out-of-roundness' of machined parts has the Weibull distribution

$$f(x) = \beta\sigma^{-1}x^{\beta-1}\exp(-x^\beta/\sigma), \quad x > 0,\ \sigma > 0,\ \beta > 0.$$

The paper is concerned with tests of the hypothesis $\sigma\leqq\sigma_0$ against the alternative $\sigma>\sigma_0$ assuming β to be known. The uniformly most powerful test is found to be based on the statistic $\sum x^\beta$. For quality control purposes, the author considers another test based on the statistic $\rho_{ij}=x^\beta_{(n+1-j)}-x^\beta_{(i)}$, where $x_{(k)}$ is the kth order statistic. This test is shown to have maximum asymptotic relative efficiency .65 for $i=0$ and $j\doteq n/5$ where by definition $x_{(0)}=0$. To judge from experiments, these results give good approximations even for rather small samples. The paper provides tables which allow the routine application of the ρ_{01}-test for quality control purposes.

G. E. Noether (Boston, Mass.)

1677:

Savage, I. Richard. A production model and continuous sampling plan. J. Amer. Statist. Assoc. **54** (1959), 231–247.

"A production model is considered where the quality of output decreases until corrective action is taken and then the cycle is repeated. For this model, the sampling plan of examining each Fth item and taking corrective action whenever a defective is found, is evaluated. Specifically, the probability of producing a good item after t units of production is assumed to be $PR^{z(t)}$, where $z(t)$ is the value of a Poisson process and represents the change in the production process since "trouble-shooting". P and R are "quality" parameters satisfying $0<P\leq 1$ and $0<R\leq 1$. Costs, such as that of looking for and removing trouble, and of inspection, are introduced. The average income is maximized by the choice of F." (From the author's summary)
D. M. Sandelius (Göteborg)

1678:

Seal, K. C. A single sampling plan for correlated variables with a single-sided specification limit. J. Amer. Statist. Assoc. **54** (1959), 248–259.

A model of sampling inspection is considered where decisions are based on observations on auxiliary variables only as soon as reliable estimates of variances and covariances of these variables and the main variable are available, all variables being assumed to be normally correlated.
D. M. Sandelius (Göteborg)

Enter under Colloque —

1679:

★Pearson, E. S. The application of the theory of probability to industrial problems. L'application du calcul des probabilités. Colloque tenu à Genève, 12–15 juillet, 1939, pp. 161–181. Collection Scientifique. Institut International de Coopération Intellectuelle, Paris, 1945. 276 pp. 10 francs suisses.

This is an expository account, written nearly a quarter of a century ago, of the state of probability theory, with especial reference to quality control, at that time. The reader with a historical inclination will doubtless be interested in a first-hand account giving one of the points of view common in the thirties with respect to topics such as "measure of belief", randomness, relative frequencies, statistical uniformity. The remarks concerning quality control are fairly standard, but the author fails to make a clear-cut distinction between acceptance sampling and hypothesis testing. Furthermore, it is astonishing to find any paper, even one written in 1939, which dichotomizes the "points of view" in probability so as to lead the reader to believe that he must select the point of view of H. Jeffreys [Proc. Roy. Soc. London Ser. A **167** (1938), 464–483] or that of J. Neyman [Philos. Trans. Roy. Soc. London Ser. A **236** (1937), 333–380]; the writings of Fisher and Student, propounding the point of view of the "unique sample", were certainly known in 1939, and have since been widely accepted as providing the only proper conceptual background for basic research. The reader interested in a deeper and more up-to-date discussion of this whole field should consult Sir Ronald Fisher's *Statistical Methods and Scientific Inference* [Oliver and Boyd, Edinburgh, 1956].
R. G. Stanton (Waterloo, Ont.)

1680:

Kapur, M. N. A property of the optimum solution suggested by Paulson for the K-sample slippage problem for the normal distribution. J. Indian Soc. Agric. Statist. **9** (1957), 179–190.

Paulson gave an optimum solution for the k-sample slippage problem under the restrictions on the parameter space in which one of the populations might have slipped to the right, as regards its mean, by a specified amount $\Delta(>0)$ while the means of the remaining populations remain equal. In this paper this restriction of only one population is relaxed. The paper shows that the Paulson procedure is unbiased in the sense that the probability of incorrect choice never exceeds the probability of correct choice among the $k+1$ decisions D_0 and D_i $(i=1, 2, \cdots, k)$, where D_0 means the decision that $m_1=m_2=\cdots=m_k$, while each D_i means the decision that $m_i=\max(m_1, m_2, \cdots, m_k)$, m_i being the population mean of the ith population distributed in the normal distribution $N(m_i, \sigma^2)$ $(i=1, 2, \cdots, k)$.
T. Kitagawa (Fukuoka)

1681:

★Gayen, A. K.; and Roy, G. C. On auto-correlations of harmonic functions. Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 345–350. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi+362 pp.

The authors discuss some elementary properties of autocorrelation of time series composed of a predominantly harmonic element. Some "misprints" ($y=t$ or $y=\sin t$; $t=2\pi/\lambda$, $\lambda=1, 2, 3, \cdots$ instead of $y=\sin(2\pi t/\lambda)$; $t=1, 2, 3, \cdots$, $\lambda=1, 2, 3, \cdots$) and incomplete explanation of the meaning of the diagrams ($x=\sin 2\pi(t+1)/\lambda$?) hampers the understanding of the simple results.
L. Törnqvist (Helsinki)

1682:

Goodman, Leo A. A note on Stepanow's tests for Markov chains. Teor. Veroyatnost. i Primenen. **4** (1959), 93–96. (Russian summary)

Let $X_1, X_2, \cdots, X_n$ be an observed sequence from a positively regular Markov chain with constant transition probability matrix (p_{ij}), where the possible states are $1, 2, \cdots, m$. Let $f_{\mathbf{u}}=f_{\mathbf{u}(s)}$ be the frequency of the s-tuple $\mathbf{u}=\mathbf{u}(s)=(u_1, \cdots, u_s)$ in the sequence, and let

$$f_{\mathbf{u},1}=f_{\mathbf{u}(s),1}=np_{u_1}'\prod_{i=1}^{s-1}p_{u_iu_{i+1}},$$

where p_i' are the stationary probabilities of the chain. Let

$$\psi_{s,1}^2=\sum_{\mathbf{u}}(f_{\mathbf{u}}-f_{\mathbf{u},1})^2/f_{\mathbf{u},1}$$

where the summation is over all the k_s values of $\mathbf{u}$ where $f_{\mathbf{u},1}>0$. M. S. Stepanow [Teor. Veroyatnost. i Primenen. **2** (1957), 143–144], generalizing some work of the reviewer [Biometrika **42** (1955), 531–533; **44** (1957), 301; MR **17**, 381], proved that $\Delta\psi_{s,1}^2=\psi_{s,1}^2-\psi_{s-1,1}^2$ and $\Delta^2\psi_{s,1}^2$ are asymptotically distributed like χ^2.

The author now points out that $\Delta^2\psi_{s,1}$ cannot be used to test the composite statistical hypothesis that the sequence is from a first order Markov chain of unspecified transition probabilities. He suggests a statistic which can

be used for this purpose. He has generalized his results in Ann. Math. Statist. **30** (1959), 154–164 [MR **21** #412].

I. J. Good (Teddington)

1683a:

Blackman, R. B.; and Tukey, J. W. The measurement of power spectra from the point of view of communications engineering. I. Bell System Tech. J. **37** (1958), 185–282.

1683b:

Blackman, R. B.; and Tukey, J. W. The measurement of power spectra from the point of view of communications engineering. II. Bell System Tech. J. **37** (1958), 485–569.

This is a survey of modern spectral analysis of stationary stochastic processes, written in the language of the communication engineer and with the emphasis on the practical use of spectral estimation. The authors discuss the reasons why the approach via the spectrum is more natural, more essential and simpler than the methods based on the correlation function.

Basic definitions and properties of spectra, covariance functions etc. are given together with a discussion of the corresponding statistical problems of estimation. Many of the spectral estimates that have been suggested are mentioned, their sampling properties are studied and computational techniques are developed.

What makes this monograph especially valuable is the emphasis the authors put on certain aspects of the problem. Sometimes the results are new, sometimes they have been given before but not discussed as carefully as here. The variability of the estimates has been studied a good deal before, but the approximations suggested here should be very useful because of their simplicity even if they are quite crude. The balance between resolvability and variability is brought out clearly. There is a long and detailed discussion of aliasing. This is the phenomenon that occurs when, sampling from a continuous time process, we observe only equidistant values. Unless these values are very close to each other the high frequencies will be superimposed upon the lower ones during the estimation procedure, and an effort must be made to correct for this or at least to warn us of possible errors. Since in practice we almost always sample from a continuous time process the problem of aliasing must be given a good deal of attention.

Another topic that will be new to most readers is prewhitening. This is a technique consisting in subjecting the observations to a linear operation before spectral analysis. The operator should have a transfer function such that its output has a more uniform (whiter) spectrum than before. The output is then analysed in the ordinary way, and the obtained spectrum sent back through the operation. Since flat spectra are easier to handle than peaked ones, we can expect a gain in information through prewhitening. Although the subject of prewhitening will require further study, the arguments that the authors advance in favor of it seem well founded.

It is claimed that the prewhitening procedure has to be only moderately successful in flattening the spectrum, but it is clear that we need some knowledge of the form of the spectrum to be able to find a suitable linear transformation. Sometimes this knowledge is given to us from earlier observations or from theoretical grounds. Otherwise a pilot sample has to be taken and analyzed. Since only low accuracy is required the authors suggest some simple and fast methods for this purpose.

It is customary to assume that the observations have mean zero to start with. In practical work this will not be the case and the authors give some advice on what to do here. Two cases are discussed: a constant mean value and a linear trend. In both cases we get trouble at the low frequencies, but we can do fairly well by simple methods.

Much attention is given to the numerical aspect, and this is certainly necessary since the computational effort is sometimes overwhelming. Again this is a reason why even crude estimates may prove useful. In this connection one should also mention the remarks on planning the measurements and the analysis for a given physical set up.

The authors give a glossary of terms, many of which are not in current use, and an appendix on Fourier technique.

Because of the special emphasis of this monograph it represents a most useful complement to the already existing literature in this field. It is recommended to anybody working in this part of statistical theory or using the spectral estimation theory in practice.

U. Grenander (Stockholm)

1684:

★**Blackman, R. B.; and Tukey, J. W. The measurement of power spectra: From the point of view of communications engineering.** Dover Publications, Inc., New York, 1959. x+190 pp. $1.85.

Except for an eight page preface and an index, the contents are identical with the article of the same title, in two parts, reviewed above.

1685:

★**Tukey, John W. The estimation of (power) spectra and related quantities.** On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 389–411. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U.S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert). $4.50.

The author reviews some topics in spectral analysis of time series. Concepts like stationarity, moments, covariance function, spectra and cross spectra are defined. He points out some of the difficulties associated with the empirical determination of a continuous spectrum and discusses aliasing, resolvability and variability. Prewhitening and computational techniques are mentioned briefly, and the single function approach of N. Wiener is compared to ensemble reasoning.

The problem suggested in section 4 on a "finite" representation of a covariance function can be answered by the principal theorem of Carathéodory on this topic; see e.g. Grenander and Szegö, *Toeplitz forms and their applications* [Univ. of California Press, Berkeley-Los Angeles, 1958; MR **20** #1349; p. 56].

U. Grenander (Stockholm)

1686:

Matoušek, Vladimír. The statistical theory of detection of radioactive disintegration. Apl. Mat. **4** (1959), 53–74. (Czech. Russian and English summaries)

The author gives the distribution laws of the number of counts in a given time interval and of the waiting time for a given number of counts under assumptions as close as possible to the real conditions of measurement. A brief account of the distribution laws concerning the decay of a single radioactive nucleus and that of the homogeneous emitter is given. Then the concept of counting efficiency is discussed. The usual assumption of equal counting probability of each nucleus is dropped and it is shown that the generalized binomial distribution of Poisson provides a suitable basis for the derivation of simple asymptotic distributions in which the total counting efficiency appears. In two theorems the stationary case is considered and in two other theorems the decrease of activity is taken into account. Certain variables are proved to be asymptotically normal. In subsequent sections the influence of imperfect resolving efficiency of the apparatus is treated for the case of recurrent processes. For two models of random dead time—the dead period following after each registration and after each primary event—the general distribution function of the waiting time between two subsequent registrations is derived. The background of the detector is considered. *J. Janko* (Prague)

NUMERICAL METHODS

See also 1284, 1315, 1316, 1388, 1661, 1670, 1793, 1798, 1845a-b, 1892, 1893.

1687:

Weisfeld, Morris. Orthogonal polynomials in several variables. Numer. Math. **1** (1959), 38–40.

The well-known recursion formula between 3 consecutive orthogonal polynomials of a real variable x is generalized to polynomials in several variables x_j and used in order to construct a set of such polynomials in a domain D of the n-dimensional Euclidean space. The construction is not trivial because special orderings of the powers of the x_j must be introduced. If D is a Cartesian product $D_1 \times D_2$ a set of orthogonal functions in D may be obtained by multiplication of functions taken from orthogonal sets in D_1 and D_2 respectively. *E. Stiefel* (Zürich)

1688:

Collatz, L.; und Schröder, J. Einschliessen der Lösungen von Randwertaufgaben. Numer. Math. **1** (1959), 61–72.

For iteration-methods of the form $u_{n+1}=Tu_n$, where T is a monotonic non-falling or non-increasing operator in a semi-ordered metric space, it has been previously shown [D. Morgenstern, Dissertation, T.U. Berlin, 1952 and J. Schröder, Z. Angew. Math. Mech. **36** (1956), 260–261; MR **18**, 337] that for a suitable choice of the initial approximation u_0 the subsequent approximations enclose a solution of the operator equation $u=Tu$.

In the present paper this result is again formulated and completely proven. The resulting theorems are then applied to boundary value problems of the form $M[u]=f(x, u)$, $U_\mu[u]=\gamma_\mu$ $(\mu=1, 2, \cdots)$, where M and U_μ are linear differential operators and x denotes a finite dimensional variable. In addition some explicit examples are discussed. *W. Rheinboldt* (Syracuse, N.Y.)

1689:

Schröder, Johann. Fehlerabschätzung bei linearen Gleichungssystemen mit dem Brouwerschen Fixpunktsatz. Arch. Rational Mech. Anal. **3** (1959), 28–44.

Consider a partially ordered m-dimensional linear space and a linear system of equations $Gu=r$ which maps this space into itself. Splitting the matrix G into a difference $G=A-B$ where A is regular, one obtains the equivalent system $u=Tu=Mu+s$ where $M=A^{-1}B$ and $s=A^{-1}r$. Consider a domain ϑ of vectors u defined by $x\leq u\leq y$ with fixed vectors x, y $(x\leq y)$. A necessary and sufficient condition suggests itself under which the operator T maps ϑ into itself. Under this assumption the Brouwer fixed-point theorem can be applied and establishes the existence of a solution u^* within ϑ and in the same moment gives error-estimates for u^* in terms of the vectors x, y, Tx, Ty, etc.

This general result is reduced to results previously proven by the author for the case of a monotonically decreasing or increasing operator T [J. Schröder, Z. Angew. Math. Mech. **36** (1956), 260–261; MR **18**, 337; and #1688 above]. Another specialization of the general theorem concerns the error estimates involving the first iterate $u_1=Tu_0$ of a given approximation u_0 of the solution u^*. The results are then applied to obtain error-estimates for the total-step and for the single-step-iteration. In addition some general remarks are made about the way in which the matrix G has to be split into the difference $G=A-B$ so that the general theorem is applicable. As a conclusion, two numerical examples are given. *W. C. Rheinboldt* (Syracuse, N.Y.)

1690:

Muller, Mervin E. An inverse method for the generation of random normal deviates on large-scale computers. Math. Tables Aids Comput. **12** (1958), 167–174.

A very fast method of generating random numbers from the normal distribution function is presented, using as starting point a generator of positive random numbers with a rectangular distribution. The method is essentially one of interpolation, linear for the most part, in a table of values of the inverse of the cumulative distribution function. The tabular intervals are selected to achieve maximum speed on a binary computer. The method sacrifices storage space for speed, in that a table of some 160 constants must be available. *B. A. Chartres* (Sydney)

1691:

Goldstein, M.; and Thaler, R. M. Bessel functions for large arguments. Math. Tables Aids Comput. **12** (1958), 18–26.

The authors note that the magnitude and angle (amplitude and phase functions) of the Hankel functions of real argument are slowly changing for large argument and so provide a convenient means for computing Bessel functions for large argument. The first six terms of the power series expansions about the point at infinity are obtained for the amplitude and phase functions from their differential equations. (It might be noted that the asymptotic expansion of the square of the amplitude function is given in G. N. Watson's treatise [*A treatise on the theory of Bessel functions*, 2nd. ed., University Press, Cambridge, 1945; MR **6**, 64; p. 449].) *R. G. Langebartel* (Urbana, Ill.)

1692:

Salzer, Herbert E. Tables for the numerical calculation of inverse Laplace transforms. J. Math. Phys. **37** (1958), 89–109.

This paper gives numerical coefficients for evaluating inverse Laplace transforms

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}F(p)dp$$

by the approximate formula

$$f(t) \approx \sum_{k=1}^{m} A_k^{(m)}(t)F(k)$$

used when $F(p)$ is considered to be of the form $\sum_{r=1}^{m} \beta_r/p^r$. The theory is discussed, and the method of use, with examples.

Tables are given for $m=1(1)6$, $t=0(.1)6$; $m=7$, 8, $t=0(.2)m$; $m=9$, 10, $t=0(1)m$. Coefficients are exact or to 8 figures.

The numerical examples given indicate considerable cancellation when summing, so that the 8 figures given are none too many.

This useful paper is a companion work to Salzer's earlier "Tables of coefficients for the numerical calculation of Laplace transforms" [Nat. Bur. Standards Appl. Math. Ser. No. 30, 1953; MR **15**, 163].

J. C. P. Miller (Cambridge, England)

1693:

Marimont, Rosalind B. A new method of checking the consistency of precedence matrices. J. Assoc. Comput. Mach. **6** (1959), 164–171.

A precedence set is a set of elements on which a transitive relation, < meaning "precedes", has been defined. Such a set may be described by a matrix in which $a_{ij}=1$ signifies that the ith element immediately precedes the jth, $a_{ij}=0$ signifies the absence of that immediate precedence. A precedence set is inconsistent (and impossible to implement) if the transitivity property implies that some element precedes itself.

The conventional method for detecting inconsistency is to raise the precedence matrix to successively higher powers. If, for example, element i precedes element j, j precedes k, and k precedes i (an inconsistency) we should have $a_{ij}=a_{jk}=a_{ki}=1$. Then, using obvious notation, in the square $a_{ik}^{(2)}=1$ and in the cube $a_{kk}^{(3)}=1$, disclosing inconsistency. More complicated inconsistencies are revealed by higher powers of the matrix; if a matrix is consistent some power will be the null matrix.

This paper proposes a less laborious method for detecting inconsistency. A consistent precedence set must have at least one entrance element—one not preceded by the other. The matrix column corresponding to an entrance element consists of zeroes. Similarly it must have some final element(s), whose rows consist of zeroes. If the row and column corresponding to an entrance element (or a final element) be deleted, the reduced matrix must still include some entrance (or final) element, whose row and column can be deleted in turn. Hence by striking the rows and columns corresponding to entrance and final elements in successively reduced matrices, either all rows and columns can be stricken or, if not, an inconsistency has been discovered. This algorithm is far simpler than raising the matrix to high powers.

R. Dorfman (Cambridge, Mass.)

1694:

Noble, B. The numerical solution of an infinite set of linear simultaneous equations. Quart. Appl. Math. **17** (1959), 98–102.

The usual method for the numerical solution of an infinite set of linear simultaneous equations is to select n equations in n unknowns and consider the solution as an approximation to that desired. The author suggests the use of the solution of m equations in m unknowns, $m=1, 2, 3, \cdots$. The increments or "corrections" to the solutions resulting from enlarging m can be obtained from the results of commonly used elimination methods. The author uses these in making estimates of the errors involved in solving only a finite number of the equations.

P. S. Dwyer (Ann Arbor, Mich.)

1695:

Dück, W. Eine Fehlerabschätzung zum Einzelschrittverfahren bei linearen Gleichungssystemen. Numer. Math. **1** (1959), 73–77.

Es wird eine Fehlerabschätzung für das Iterationsverfahren in Einzelschritten bei linearen Gleichungssystemen mit überwiegender Hauptdiagonale angegeben, welche weniger scharf, aber einfacher als eine von Sassenfeld [Z. Angew. Math. Mech. **31** (1951), 92–94; MR **14**, 692] bewiesene Abschätzung ist.

J. Schröder (Hamburg)

1696:

Chao, F. H. Comparison of gradient methods with applications. Acta Math. Sinica **7** (1957), 63–78. (Chinese. English summary)

A translation of this paper is reviewed below.

1697:

Chao, F. H. Comparison of gradient methods with applications. Sci. Sinica **7** (1958), 565–581.

The paper is divided into two parts. In the first part is a class of gradient methods for solving a system of linear equations $Ax+b=0$. These could be unified by introducing a suitable norm. They are extended to the case in which two parameters are used instead of one. This yields an acceleration device. In the second part is a gradient method for finding eigenvalues of a symmetric matrix A. The function to be diminished is the quantity

$$|x|^2|Ax|^2-(x, Ax)^2,$$

under the assumption that $A>0$. Numerical examples comparing various methods are given.

M. R. Hestenes (Los Angeles, Calif.)

1698:

Gastinel, Noël. Conditionnement d'un système d'équations linéaires. C. R. Acad. Sci. Paris **248** (1959), 2707–2709.

Let $x=(\xi_1, \cdots, \xi_n)$ be a n-dimensional vector with ξ_i real and let

$$\|x\| = (\sum \xi_i^2)^{1/2}, \quad \Phi(x) = \sum|\xi_i|.$$

Let A be a real non-singular $n\times n$ matrix. Let m_A, M_A denote the extrema, for $x\neq 0$, of $\Phi(Ax)/\|x\|$. Let

$$N(A) = \{\sum a_{ij}^2\}^{1/2}, \ \mu_0(A) = \min_i\Big(\sum_j a_{ij}^2\Big)^{1/2}.$$

It is stated that $(m_A/M_A)\leq\mu_0(A)/N(A)$ with equality if

and only if A is orthogonal. This suggests that $C_A = (m_A N(A)/M_A \mu_0(A))$ should be used as a condition number: $C_A = 1$ is equivalent to orthogonality and $C_A = 0$ to singularity. Various properties of this condition number are noted and its values in special cases are given.

John Todd (Pasadena, Calif.)

1699:

Ghosh, P. K. A note on the solution of linear equations by the method of least squares. Bull. Calcutta Math. Soc. **50** (1958), 19–22.

The solution by least squares under discussion is that of the problem in which there are n linear equations (each having a weight w_i) in s unknowns with $s < n$ and with the equations not strictly compatible with each other. A geometrical interpretation is given to the method so that it is applicable to certain linear functional equations in Hilbert space.

P. S. Dwyer (Ann Arbor, Mich.)

1700:

Wilkinson, J. H. The calculation of eigenvectors by the method of Lanczos. Comput. J. **1** (1958), 148–152.

The author discusses the method of Lanczos [J. Res. Nat. Bur. Standards **45** (1950), 255–282; Proc. Sympos. on Spectral Theory and Differential Problems, pp. 301–316, Oklahoma Agric. and Mech. College, Stillwater, Okla., 1951; MR **13**, 163, 497] with special attention given to the instability of the method which tends to destroy the orthogonality (symmetric case) of the iteration vectors. As a remedy he proposes essentially a combination with a Schmidt-orthogonalisation process. In the nonsymmetric case, an analogeous bi-orthogonalisation is applied. It is interesting to see that this does in no way disturb the determination of the roots as well as the eigenvectors from the $\alpha\beta$-matrix.

Furthermore the author treats the singular case, where the Lanczos method ends prematurely because some iteration vector vanishes; in such a case one simply continues with a new vector chosen orthogonal to all previous iteration vectors but otherwise arbitrarily. Here the corresponding nonsymmetric case is more complicated since the two biorthogonal systems b_i, b_i^* of iteration vectors may behave differently; for instance for some k we may have $b_k = 0$ whereas $b_k^* \neq 0$. The remedy proposed for this occurrence is quite unusual but obviously works. The author points out, however, that the nonsymmetric case is very delicate with respect to roundoff errors and therefore proposes to use double accuracy throughout.

It may be noted, that many of the items discussed by the author have already been treated by the reviewer [Z. Angew. Math. Physik **4** (1953), 35–56; MR **14**, 1055].

H. Rutishauser (Zürich)

1701:

Gleyzal, André N. Solution of non-linear equations. Quart. Appl. Math. **17** (1959), 95–96.

It is desired to solve the system of equations: $f_i(x_j) = 0$, $i, j = 1, 2, \cdots, N$. If $\bar{x}_j$ is an approximate solution, it is common to seek a better solution in the form $\bar{x}_j + d_j$ where d_j satisfies the linear system

$$\bar{f}_i + \sum_j \frac{\partial \bar{f}_i}{\partial x_j} d_j = 0.$$

Bars indicate that the corresponding quantities are evaluated at $\bar{x}_j$. The author however chooses $\bar{x}_j + \lambda' d_j$ as the next approximation to the solution, where λ' is chosen so that

$$\varphi(\lambda) \equiv \sum [f_i(\bar{x}_j + \lambda d_j)]^2 \quad (\lambda \text{ real})$$

has a relative minimum at $\lambda = \lambda'$. The author states that the method has been found to work well in various cases where standard methods (including "steepest descent") were unsatisfactory.

M. A. Hyman (Yorktown Heights, N.Y.)

1702:

Campbell, Edwin S.; Fishbach, E. M.; and Hirschfelder, J. O. Coefficients and roots of polynomials which define the derivatives of the exponential of $(-\varepsilon/T)$. Math. Tables Aids Comput. **12** (1958), 1–17.

Letting $x = -\varepsilon/T$ and writing the nth derivative of e^x as

$$d^n(\exp x)/dT^n = T^{-n} e^x W_n(x), \quad W_n(x) = \sum_{k=1}^{n} b_{n,k} x^k,$$

the authors furnish tables giving $b_{n,k}$ and the zeros of $W_n(x)$ for $0 < n \leq 30$. The $W_n(x)$ may be expressed in terms of the Laguerre polynomials $L_n(x)$ as

$$W_n(x) = (-1)^n [L_n(x) - n L_{n-1}(x)].$$

M. Marden (Milwaukee, Wis.)

1703:

Salzer, Herbert E.; and Kimbro, Genevieve M. Extension of Lindow's tables for numerical differentiation using Newton-Stirling and Newton-Bessel differences. Math. Tables Aids Comput. **12** (1958), 133–140.

This paper tabulates coefficients in the four formulas expressed in central difference notation

$$hf'(x_0 \pm ph) = \mu\delta_0 \pm p\delta_0^2 + \sum_{m=1}^{n} \{A_{2m+1}(p)\mu\delta_0^{2m+1} \pm A_{2m+2}(p)\delta_0^{2m+2}\} + R_{1,n}$$

$$h^2 f''(x_0 \pm ph) = \delta_0^2 \pm p\mu\delta_0^3 + \sum_{m=1}^{n} \{C_{2m+2}(p)\delta_0^{2m+2} \pm C_{2m+3}(p)\mu\delta_0^{2m+3}\} + R_{3,n}$$

and, writing $p = \frac{1}{2} \pm p_1$,

$$hf'(x_0 + ph) \equiv hf'(x_0 + \tfrac{1}{2}h \pm p_1 h) = \delta_{1/2} \pm p_1 \mu\delta_{1/2}^2 + \sum_{m=1}^{n} \{B_{2m+1}(p_1)\delta_{1/2}^{2m+1} \pm B_{2m+2}(p_1)\mu\delta_{1/2}^{2m+2}\} + R_{2,n}$$

$$h^2 f''(x_0 + ph) \equiv h^2 f''(x_0 + \tfrac{1}{2}h \pm p_1 h) = \mu\delta_{1/2}^2 \pm p_1 \delta_{1/2}^3 + \sum_{m=1}^{n} \{D_{2m+2}(p)\mu\delta_{1/2}^{2m+2} \pm D_{2m+3}(p)\delta_{1/2}^{2m+3}\} + R_{4,n}.$$

Explicit polynomial forms are given for coefficients of differences to the tenth, with 10-figure numerical values of these coefficients for p or $p_1 = 0(.01).25$.

The tables extend those of M. Lindow, *Numerische Infinitesimalrechrung* [Dümmler, Berlin, 1928], pp. 166–169.

J. C. P. Miller (Cambridge, England)

1704:

Szidarovszky, J. Eine praktische Methode zur Lösung von linearen Differentialgleichungen mit nicht konstanten Koeffizienten. Acta Tech. Acad. Sci. Hungar. **24** (1959), 85–94. (English, French and Russian summaries)

Bei der hier angegebenen Methode zur Lösung von Randwertaufgaben bei gewöhnlichen Differentialgleichungen wird die gegebene Differentialgleichung durch eine Differentialgleichung mit stückweise konstanten Koeffizientenfunktionen ersetzt. Das Ziel ist, diese Ersatzdifferentialgleichung auf möglichst einfache Weise zu lösen. Hier wird vorgeschlagen, eine für das gesamte Intervall gültige allgemeine Lösung der Form

$$y = A_1 g_1(x) + \cdots + A_n g_n(x) + g_0(x)$$

zu ermitteln, und es wird ein Weg zur Berechnung der $g_i(x)$ angegeben. Numerisches Beispiel.

J. Schröder (Hamburg)

1705:

Grosswald, Emil. Transformations useful in numerical integration methods. J. Soc. Indust. Appl. Math. 7 (1959), 76–84.

The solution of the set of linear simultaneous equations

$$\dot{x} = Ax + f(t)$$

(x a vector, A a matrix), subject to the initial condition $x = x_0$ when $t = 0$ may be written as

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}f(\tau)d\tau.$$

This leads to an obvious recurrence relation between successive x_n (x_n is value of $x(t)$ when $t = nh$), viz:

$$x_{n+1} = e^{Ah}x_n + F(n),$$

where

$$f(n) = e^{A(n+1)h}\int_{nh}^{(n+1)h} e^{-A\tau}f(\tau)d\tau.$$

$F(n)$ takes a simple form if $f(\tau)$ can be expanded in a finite or absolutely convergent infinite Fourier series.

The author considers simple integration formulae, in particular the Euler extrapolation method, and shows that, in view of the recurrence relation above, it is possible to transform the equations $\dot{x} = Ax + f(t)$ to the form $\dot{y}_n = By_n + g(n)$ where $y_n = y(nh)$ which may be solved exactly.

Some remarks on the number of calculations required by the method are of interest. *D. C. Gilles* (Glasgow)

1706:

Dahlquist, Germund. Stability and error bounds in the numerical integration of ordinary differential equations. Kungl. Tekn. Högsk. Handl. Stockholm. No. 130, 87 pp. (1959).

In this thesis the author considers systems of differential equations

(1) $d^r y/dx^r = f(x, y), \quad y^{(\rho)}(a) = y_0^{(\rho)} \quad (\rho = 0, \cdots, r-1)$

[$y = (y^i)$, $f = (f^i(x, y^1, \cdots, y^s))$ column vectors of dimension s] and studies corresponding difference methods of the form

(2) $$\sum_{\kappa=0}^{k} (\alpha_\kappa y_{n+\kappa} - h^r\beta_\kappa f_{n+\kappa} + h^{r+1}\gamma_\kappa f'_{n+\kappa}) = 0,$$

where α_κ, β_κ, γ_κ are real constants, $\alpha_k > 0$, $|\alpha_0| + |\beta_0| > 0$, $x_m = a + mh$, $f_m = f(x_m, y_m)$, $f_m' = [\partial f/\partial x + (\partial f/\partial y)f]_{x=x_m, y=y_m}$, $(\partial f/\partial y)$ the Jacobian matrix $(\partial f^i/\partial y^j)$. It is thereby assumed that all $\gamma_\kappa = 0$ if $r > 1$.

With the difference equation (2) is associated the linear operator

$$Lu(x) = \rho(E)u(x) - h^r\sigma(E)u^{(r)}(x) + h^{r+1}\tau(E)u^{(r+1)}(x),$$

where $u(x)$ is a scalar function, $Eu(x) = u(x+h)$ and

$$\rho(\zeta) = \sum_{\kappa=0}^{k} \alpha_\kappa\zeta^\kappa, \quad \sigma(\zeta) = \sum_{\kappa=0}^{k} \beta_\kappa\zeta^\kappa, \quad \tau(\zeta) = \sum_{\kappa=0}^{k} \gamma_\kappa\zeta^\kappa.$$

k is called the order of L. The degree p of L is the largest integer p' such that $Lw(x) \equiv 0$ for polynomials w of degree $\leq p' + r - 1$. If $u(x) \in C^{p+r}$, then $Lu(x) = O(h^{p+r})$. A stable operator L is one for which $p \geq 1$ and all zeros of $\rho(\zeta)$ are within or on the unit circle, the zeros on the unit circle having multiplicity $\leq r$.

Extending a theory developed earlier for the case $r = 1$; $\tau(\zeta) \equiv 0$ [Math. Scand. 4 (1956), 33–53; MR 18, 338], the author shows that stability of L is equivalent to "stable convergence", as $h \to 0$, of appropriate solutions of (2) towards the solution of (1). Lack of stability, called "strong instability", renders difference methods (2) useless for practical purposes. Concerning stable operators L the following two theorems, among others, are proved. Theorem: The largest possible degree among all stable operators L of order k, with $\tau(\zeta) \equiv 0$, is less than or equal to $2([k/2] + [(r+1)/2])$. Equality holds, e.g., if $r \leq 2$, or if $r = 3$ and $k \geq 6$. Theorem: If $r = 1$, the largest possible degree among all stable operators L of degree k is equal to $k + 2$.

Instabilities of a weaker type, caused by zeros ζ_j of $\rho(\zeta)$ with $|\zeta_j| = 1$, $\arg\zeta_j \neq 0$, are also studied. For operators L with $r = 1$, $\tau(\zeta) \equiv 0$, it is shown that "weak instabilities" are particularly notable if $\Re g(\lambda_j) < 0$, where $g(\zeta) = \sigma(\zeta)/(\zeta\rho'(\zeta))$. Theorem: If L is a stable operator of even order k and maximal degree $k+2$ and, consequently, $\zeta_j = e^{i\theta_j}$, $0 = \theta_1 < \theta_2 < \cdots < \theta_k < 2\pi$, $\theta_j = \pi$ for some $j = j_0$, then the sequence $g(\lambda_j)$ $(j = 1, 2, \cdots, k)$ is real, alternating in sign for $j > 1$, and $g(\lambda_{j_0}) \leq -1/3$.

A substantial part of this work is concerned with estimating the norm $\|y_n - y(x_n)\|$ of the error vector. Bounds are obtained from the linear variational equation $dz/dx = B(x)z$, $B(x) = (\partial f/\partial y)_{y=y(x)}$. To make up for the linearization, certain correction terms are introduced and estimated separately. A feature of these error formulas is the appearance of a directional derivative $\mu[B(x)] = \lim_{t\to 0+} t^{-1}(\|I + tB(x)\| - 1)$ in place of the usual norm $\|B(x)\|$, which renders the bounds more realistic if $\mu[B(x)] < 0$.

Finally a theory of stability is sketched for the case where a system of two difference equations of the form (2) is applied to integrate $y'' = f(x, y, y')$.

Walter Gautschi (Oak Ridge, Tenn.)

1707:

Varga, R. S. Numerical solution of the two-group diffusion equation in $x-y$ geometry. Trans. I. R. E. NS-4 (1957), 52–62.

The author analyzes theoretically a very efficient method developed by him for solving the two-group diffusion equations of nuclear reactor theory. First, he derives the usual difference approximations to these equations; in matrix notation, they can be written

(*) $$A_1\phi = B_1\psi, \quad A_2\psi = \eta B_2\phi,$$

Φ and Ψ representing the slow and fast fluxes, respectively. If one suppresses components corresponding to zero slow flux Φ, then the A_i are symmetric and positive definite, with positive diagonal elements and nonpositive off-diagonal elements; the B_i are non-negative diagonal matrices. One can combine the two equations (*) into a single equation $A_1\Phi=\eta B_1A_2^{-1}B_2\Phi$, or $\Phi=\eta T\Phi$. He shows that simple iteration of the transformation $\Phi_{i+1}=T\Phi_i/\|T\Phi_i\|$ converges to a unique positive vector Φ_∞, and the scalars $\|T\Phi_i\|/\|\Phi_i\|$ converge to the associated eigenvalue $1/\eta$; the proof is based on theorems of Perron and Frobenius. Equations (*) can be solved by David Young's systematic overrelaxation method [Trans. Amer. Math. Soc. **76** (1954), 92–111; MR **15**, 562]. The best overrelaxation factors ω_1, ω_2 can be estimated by min-max inequalities involving a positive parameter α; the technique for doing this is illustrated by a numerical example.

G. Birkhoff (Cambridge, Mass.)

1708:

Hersch, Joseph. Contribution à la méthode des équations aux différences. Z. Angew. Math. Phys. **9a** (1958), 129–180.

This paper systematically applies to various problems in linear differential equations some ideas contained in the author's earlier Comptes Rendus notes [C. R. Acad. Sci. Paris **243** (1956), 1475–1478; **244** (1955), 299–302; MR **18**, 579, 678].

The first of these is to obtain a finite difference equation and boundary conditions which are satisfied by every solution of a given boundary value problem for a differential equation.

A second idea is to find a finite difference equation and boundary conditions satisfied by certain averages of the solution of a given boundary value problem. The difference equation is found to be the same as above, but the boundary conditions depend upon the particular averages used.

A third idea is to find a finite difference problem for each of a sequence of successive refinements of a finite difference grid having the property that the values of the solution for a certain grid agree at the points of this grid with the solutions for any refinement of the grid. Moreover, the solution of a given differential equation boundary value problem is to satisfy the finite difference equations with arbitrary accuracy as the grid becomes finer.

All these ideas are applied directly to ordinary differential equations whose solutions are known. The results are then used as guides to construct good finite difference approximations for other ordinary and for partial differential equations. The methods are closely related to the Mehrstellenverfahren of L. Collatz [*Numerische Behandlung von Differentialgleichungen*, 2te Aufl., Springer, Berlin-Gottingen-Heidelberg, 1955; MR **16**, 962].

Numerous numerical applications, notably to the deflections and vibration frequencies of plates and membranes of various shapes, are given.

H. F. Weinberger (College Park, Md.)

1709:

Sugai, I. Numerical solution of Laplace's equation, given Cauchy conditions. IBM J. Res. Develop. **3** (1959), 187–188.

In certain physical or engineering problems (e.g., subsonic flow behind a shock wave, design of Pierce-type electron guns) one is led to solving elliptic equations, such as $\nabla^2\phi=0$, with both ϕ and its normal derivative specified along an open boundary (Cauchy conditions). When a numerical solution is attempted using one of the standard difference approximants as a recursion relation, it is found that small errors entering the calculation grow exponentially as one "steps" across the region. The author discusses Laplace's equation, using the standard five-point equal-spacing difference approximant; he obtains ϕ on two "starting lines" from Taylor expansions valid near the boundary and then proceeds recursively. He gives the following bound for the error:

$$|\varepsilon_{m,i}|\leq 3(5.7)^{m-2}|\varepsilon_{2,i}|,$$

where $\varepsilon_{m,i}$ is the error at the ith point on the mth line. The exponent is erroneously printed as $m-3$ in the paper.

M. A. Hyman (Yorktown Heights, N.Y.)

1710:

Ogura, Yoshimitsu. On the truncation error which arises from the use of finite differences in the Laplacian operator. J. Meteorol. **15** (1958), 475–480.

Let $\phi(x,y)$ be a random isotropic function—i.e. a member of an ensemble whose statistical properties are invariant under rotation and reflection. Let $\nabla^2\phi$ and $\nabla_a{}^2\phi$ be respectively the Laplacian of ϕ and its finite difference analog (with mesh-width a). If the power spectrum of ϕ is $F(K)$, K the wave number, then the power spectra of $\nabla^2\phi$ and $\nabla_a{}^2\phi$ are respectively $K^5F(K)$ and $K^5F(K)\psi(Ka)$. Here $\psi(Ka)$ depends on the particular difference analog used. By insisting that $\psi(Ka)\simeq 1$ for small K ("long waves") the author (a meteorologist) derives weights for the points in the regular 9-point and 13-point stars, differing from the usual weights obtained by Taylor series expansion. He concludes that using a star with more than 5 points reduces the truncation error in computing $\nabla^2\phi$ (measured by $\psi(Ka)-1$) but increases the probable random error.

M. A. Hyman (Yorktown Heights, N.Y.)

1711:

Hyvärinen, L. Fourier analysis, a new numerical method. Acta Polytech. **248** (1958), 19 pp.

This paper describes a numerical procedure for evaluating the Fourier transform or Fourier series coefficients of an empirical function. The theory upon which the method is based involves certain operators of the type of the Dirac delta function. Since summations at unequally spaced points are necessary, mechanical templates may be used to facilitate the computation if it is to be done by hand or with the aid of a desk calculator. This of course is only possible when the function to be analysed is given in graphical form. Similar methods may also be used to obtain the inverse transform or to find the function from its Fourier series.

B. Lepson (Washington, D.C.)

COMPUTING MACHINES

See 1690, 1713, 1901, 1925, 1926.

MECHANICS OF PARTICLES AND SYSTEMS

See also 1421, 1433, 1833, 1929.

1712:

Forte, Bruno. Di alcune notevoli proprietà differenziali dei moti rigidi di rotolamento. Ann. Scuola Norm. Sup. Pisa (3) **12** (1958), 417–423.

Geometrical considerations on kinematics in three-dimensional space. The author supposes that the two axoids roll on another without slip. He introduces the principal plane, this being the tangent plane through the common generator at the point at infinity. For the solid motion, if restricted to the points of the principal plane, there are always infinitely many osculating plane motions. By means of these concepts a necessary and sufficient condition is given for the existence at a given instant of a inflexion point of the nth order in a trajectory.

O. Bottema (Delft)

1713:

Walther, A. und Schappert, H. Numerische Behandlung des Gelenkvierecks. Numer. Math. **1** (1959), 110–120.

Die rechtwinkligen Koordinaten eines beliebigen Punktes K der bewegten Koppelebene werden als Funktionen des Antriebwinkels betrachtet, wobei die Seitenlängen des Vierecks und die relativen Koordinaten von K als Parameter erscheinen. Die Funktionen sind doppeldeutig; ein gewisses Gebiet in der Nähe des Nullwertes von einer Hilfsgrösse wird ausgeschlossen. An Hand der gefundenen Formeln wird ein Rechenplan in algorithmischer Schreibweise aufgestellt, welcher durch ein Programm für einen elektronischen Rechenautomaten erprobt ist. Fünf Fälle sind durchgerechnet und die Bilder der bezüglichen Koppelkurven werden beigegeben. Die Methode gibt Möglichkeiten zur Berechnung von Geschwindigkeiten und Beschleunigungen.

O. Bottema (Delft)

1714:

Půst, Ladislav. Wirkung der Eigenschaften des Wechselkrafterregers auf die Schwingungen eines mechanischen Systems. Apl. Mat. **3** (1958), 428–450. (Russian. Czech and German summaries)

In dem Artikel ist die Bewegung des Schwingungssystems untersucht, das mit Hilfe der Feder und der excentrischen Scheibe, die auf der Welle des Motors befestigt ist, erzeugt wird. Die Gleichungen der Bewegung sind der Form

$$\frac{d^2y}{d\tau^2}+\varepsilon\delta\frac{dy}{d\tau}+y=\varepsilon\kappa\cos\varphi, \tag{1}$$

$$\frac{d^2\varphi}{d\tau^2}=\varepsilon(\mu_0-\mu_1)+\varepsilon^2\kappa\rho\sin\varphi(\cos\varphi-y),$$

wo ε ein kleiner Parameter ist. Zu der Lösung wird die asymptotische Methode benutzt [siehe Bogolyubov und Mitropolskiĭ, *Asimptotičeskie metody v teorii nelineĭnyh kolebaniĭ*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955; MR **17**, 368].

M. Zlámal (Brno)

1715:

Vejvoda, Otto. Eine Bemerkung zu dem Artikel von Ladislav Půst: Wirkung der Eigenschaften des Wechselkrafterregers auf die Schwingungen eines mechanischen Systems. Apl. Mat. **3** (1958), 451–460. (Russian. Czech and German summaries)

Der im Titel eingeführte Artikel ist der obenbesprochene. Der Verfasser führt seine Resultate über die Existenz und Stabilität einer stationären Lösung des Systems (1) aus der vorangehenden Rezension ein. Die Lösung wird in der Form $x=x(t,\varepsilon)$, $\varphi=\omega(\varepsilon)\cdot t+\varepsilon F(t,\varepsilon)$ $(F(0,\varepsilon)=0)$ gesucht, wo $x(t,\varepsilon)$ und $F(t,\varepsilon)$ periodische Funktionen mit der Periode $T(\varepsilon)=2\pi/\omega(\varepsilon)$ sind. Ausführliche Resultate mit Beweisen werden in einem anderen Artikel publiziert werden.

M. Zlámal (Brno)

1716:

Obmoršev, A. N. Investigation of the structure of the equations of the disturbed motion of a system in connection with the possibility of producing selfoscillations. Trudy Inst. Mašinoved. Sem. Teorii Mašin i Mehanizmov **16** (1957), no. 64, 5–25. (Russian)

The author first reviews some well-known definitions and theorems on the stability of motion. Using results of Lagrange-Dirichlet, Liapunov, and Thompson and Tait, he discusses the stability and vibrations of a rotor. The influence on stability of various types of forces is summarized.

J. P. LaSalle (Baltimore, Md.)

1717:

Osiński, Zbigniew. Forced vibration of a system of one degree of freedom due to periodic forces, with damping characterized by a strong non-linearity. Arch. Mech. Stos. **11** (1959), 33–44. (Polish and Russian summaries)

The author shows that the equation

$$\ddot{x}+k\dot{x}+\psi(\dot{x})+\omega^2x=f(t),$$

where ψ is nonlinear, has bounded solutions tending asymptotically toward a periodic one provided that the solution of the corresponding equation in which $\psi\equiv 0$ is suitably bounded.

E. Pinney (Berkeley, Calif.)

1718:

Ishlinskii, A. Iu. On the theory of complicated systems of gyroscopic stabilization. J. Appl. Math. Mech. **22** (1958), 493–512 (359–375 Prikl. Mat. Meh.).

The author states that the second method of Lagrange, which is usually used for the derivation of the equations of motion of gyroscopes, can be quite cumbersome for complicated systems. The equations of motion of such devices can be obtained in a relatively simple way, however, by the successive application of the principle of angular momentum. The description of this method, applied to a particular system of stability control, constitutes the content of this article.

H. P. Thielman (Ames, Iowa)

1719:

Rumiantsev, V. V. On the stability of motion of a gyroscope on gimbals. J. Appl. Math. Mech. **22** (1958), 513–520 (374–378 Prikl. Mat. Meh.).

The author calls attention to the known fact that the stability of the gyroscope axis in vertical position depends on both the magnitude and direction of the angular velocity vector of the outer ring. The object of the paper is to investigate this special case. For that purpose the author constructs the Liapunov function in the form of a

linear combination of the first integrals of the equations for the stability of motion of the regular precession of a gyroscope on gimbals. This yields as a special case the necessary condition for stability of the vertical position of the gyroscope axis. The influence of the dissipative forces on the stability of motion of a gyroscope is also investigated. *H. P. Thielman* (Ames, Iowa)

1720:

Chetaev, N. G. On a gyroscope mounted in a universal suspension [on gimbals]. J. Appl. Math. Mech. **22** (1958), 521–525 (379–381 Prikl. Mat. Meh.).

The author considers a gyroscope in a universal suspension (on gimbals). He assumes that the ellipsoid of inertia of the gyroscope about the fixed point is an ellipsoid of revolution. He writes out Lagrange's equations, and the differential equations for the Euler angles. He discusses the hyperelliptic integral which occurs in the solution of one of the latter equations and makes a comparison of the obtained results and Lagrange's motion. *H. P. Thielman* (Ames, Iowa)

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 1635, 1707, 1800, 1825, 1914.

1721:

Forte, Bruno. Proprietà ricorrenti del moto non stazionario di un fluido e relativa estenzione ad un numero qualunque di dimensioni. Ann. Scuola Norm. Sup. Pisa (3) **12** (1958), 397–416.

The paper is concerned with Poincaré's recurrence theorem in statistical mechanics and with the ergodic hypothesis about the equality of time- and phase-averages. The author considers the theorem and its generalization for non-stationary motions in phase-space, with critical remarks on the work of Loève [*Probability theory*, Van Nostrand, New York-Toronto-London, 1955; MR **16**, 598; Ch. IX] and gives his own views. *O. Bottema* (Delft)

1722:

Morita, Tohru. Theory of classical fluids: hyper-netted chain approximation. I. Formulation for a one-component system. Progr. Theoret. Phys. **20** (1958), 920–938.

An extensive classification of Mayer graphs of classical gas theory (virial expansion) is defined and corresponding partial summations are found possible for the partition function and the pair distribution function. *L. Van Hove* (Utrecht)

1723:

Alkemade, C. T. J. On the problem of Brownian motion of non-linear systems. Physica **24** (1958), 1029–1034.

The problem of finding the frequency spectrum of the current fluctuations in a non-linear resistor has been studied by D. K. C. MacDonald [Phys. Rev. (2) **108** (1957), 541–545; MR **20** #1450] and the reviewer [ibid. **110** (1958), 319–323; MR **20** #4941], with different results. A special example is here calculated explicitly, namely a circuit consisting of a diode at temperature T and a condenser. The noise is due to fluctuations in the thermal emission of electrons. The probability for an electron to be emitted depends exponentially on the instantaneous voltage (Richardson's formula), but the exponent is expanded to second order ("modest non-linearity"). The time of flight between the two electrodes is neglected. With these assumptions the average charge on the condenser is found to be $-\frac{1}{2}e$, in agreement with MacDonald, but the result for the high-frequency limit of the spectrum differs from the general formulas of both MacDonald and the reviewer. These discrepancies are discussed, but a final solution of the problem is still to be found. *N. G. van Kampen* (Utrecht)

1724:

★Montroll, Elliott W. The application of the theory of stochastic processes to chemical kinetics. Advances in chemical physics, Vol. I, edited by I. Prigogine, pp. 361–399. Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London; 1958. xi+414 pp. $11.50.

The authors concern themselves with the study of the departure from equilibrium in unimolecular chemical (dissociation) reactions, specifically the effect of this departure on the "microscopic" energy level distribution function of the reactant species and on the rate of activation of the reaction. After a short review of previous attempts to modify the standard equilibrium theories (collision theory, absolute rate theory) they introduce a fairly general microscopic model of the chemical reaction which is a discrete energy level (quantum-mechanical) analog of the Kramers-Brownian-motion model and also a generalization to N levels of the Zwolinsky-Eyring 4-level model.

An ensemble of reactant molecules with quantized energy levels, initially in a Maxwell-Boltzmann distribution appropriate to the temperature T_0, is immersed in a large excess of chemically inert gas which acts as a heat bath at constant temperature $T>T_0$. By collision with the heat bath, reactant molecules are step-wise excited into their higher-energy levels until at level $(N+1)$ they are irreversibly removed from the system (i.e. the reactant molecule has then decomposed into the product of the reaction). The stochastic process considered by these authors is thus a one-dimensional random walk with an absorbing barrier (at level $(N+1)$) in which the evolution in time of the fraction of molecules in a given energy level is assumed to satisfy a "master-type" transport equation. An elegant general formulation of the mean first passage time $\bar{t}$ is presented. The rate of activation is inversely proportional to $\bar{t}$ and $\bar{t}$ gives the time lag for activation of the ensemble of reactants.

Explicit, detailed calculations of the molecular fractions, distribution of mean first passage times, $\bar{t}$ and the chemical rate constant are presented only for the case where the reactants are considered as simple harmonic oscillators interacting weakly with the heat bath with Landau-Teller transition probabilities (per unit time) between adjacent energy levels. On the basis of this model it is found, in agreement with previous results on different models by other authors, that the rate of chemical reaction deviates from the equilibrium rate about 20 per cent when $E_{act}/kT=5$ and less than 10 per cent when $E_{act}/kT=10$, where E_{act} is the activation energy of the reaction and k Boltzmann's constant. Unfortunately the calculated rate of activation of unimolecular dissociation is still several orders of magnitude smaller than experimentally observed

rates. It is suggested that the answer to this classical problem of chemical kinetics may lie in a recalculation of the energy-level transition probabilities taking into account strong interactions in highly energetic collisions which lead to direct dissociation.

H. L. Frisch (Murray Hill, N.J.)

1725:

Lee, T. D.; and Yang, C. N. Many-body problem in quantum statistical mechanics. I. General formulation. Phys. Rev. (2) **113** (1959), 1165–1177.

A treatment is given of the quantum statistical partition function which is similar to Ursell's treatment in classical statistical mechanics. The new U-functions appearing in this formulation are related to the classical $\bar{U}$-functions. The classical U-functions are expressed in terms of a so-called binary kernel which can be computed from a solution of the two-body problem. The hard-sphere gas is considered in detail as an example.

D. ter Haar (Oxford)

1726:

★Biot, M. A. Linear thermodynamics and the mechanics of solids. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 1–18. American Society of Mechanical Engineers, New York, 1958. xxvii + 864 pp. $20.00.

The paper develops the general theory of linear dissipative systems from the point of view of non-equilibrium thermodynamics, under the assumption that the departure from equilibrium is small. Two fundamental functions, the free energy and the dissipation function, form the basis of the treatment. An equation describing the response of the system to arbitrary time-dependent perturbations is established. The theory is applied to thermoelasticity (including the classical theory of elasticity, the effect of temperature distribution on thermal stresses in elastic bodies, the theory of heat conduction and heat transfer, and the coupled interaction between the deformation and the temperature field which results in thermoelastic damping), to thermoelastic dissipation (i.e. thermoelasticity coupled with dynamics and thus including inertia terms), viscoelasticity and viscoelastic models, and dynamics and stress analysis of viscoelastic structures. The list of applications puts into evidence the extremely wide scope of the treatment. An extensive bibliography increases further the usefulness of the paper.

B. Gross (Rio de Janeiro)

1727:

★Eisenschitz, R. Statistical theory of irreversible processes. Oxford Library of the Physical Sciences. Oxford University Press, London, 1958. viii + 84 pp. $2.00.

The subject of this small monograph is the microphysical theory of irreversible processes in gases, liquids and solids. After a short review of equilibrium statistical mechanics, Kirkwood's 'time-smoothing' is introduced. (The author calls it 'coarse graining', although he defines coarse graining in a different way.) The resulting Boltzmann equation is used to compute the coefficients of viscosity and of thermal conductivity in dilute gases. The resulting Fokker-Planck equation is applied to liquids. Subsequently the quantum mechanical theory is cast in a form very similar to the classical theory by means of the Wigner distribution function, and the Uehling-Uhlenbeck equation is given. The solid state and its heat conduction through phonons are briefly treated. Finally, the elementary theory of Brownian movement and stochastic processes is given.

The exposition is brief, skipping many algebraic manipulations, and at times takes the form of a review paper. In this way the logical structure of the theory is made very clear, but the reading will not be easy for tiros. In making the basic step from reversible to irreversible equations the author follows mainly Kirkwood, and is accordingly rather optimistic. The equation $f^{(2)}(1, 2) = f^{(1)}(1)f^{(1)}(2)$ is introduced without comment. The criterion for the validity of the Fokker-Planck equation is stated incorrectly. In the quantum mechanical part the occurrence of irreversibility is attributed to external perturbation, which is incorrect, and not comparable with the coarse graining of classical mechanics, as the author asserts. On the other hand, there are a number of interesting remarks, which, together with the clear and coherent development of the theory, make the book worth reading.

N. G. van Kampen (Utrecht)

1728:

Nakajima, Sadao. On quantum theory of transport phenomena: steady diffusion. Progr. Theoret. Phys. **20** (1958), 948–959.

A formula of the Kubo type expressing the diffusion coefficient in terms of an equilibrium correlation is derived by the following method. One tries to find a density matrix ρ satisfying approximately the time independent Schrödinger equation $[H, \rho] = 0$ (H hamiltonian) and giving a prescribed nonuniform mass density in space, the gradient of this density being very small. For the ρ thus selected, one calculates the flow of matter. This flow depends linearly on the density gradient, the coefficients giving the diffusion tensor. The Einstein relation between diffusion and electric conductivity tensors is discussed.

L. Van Hove (Utrecht)

ELASTICITY, PLASTICITY

See also 1421, 1708, 1726, 1802, 1833, 1929.

1729:

Polozii, G. M.; and Čemeris, V. S. On the application of p-analytic functions in the axisymmetrical theory of elasticity. Dopovidi Akad. Nauk Ukrain. RSR **1958**, 1284–1287. (Ukrainian. Russian and English summaries)

Following the Goursat-Kolosov-Muskhelishvili method of application of functions of complex variable to the theory of elasticity, the author derives integral equations for the case of axially symmetric problems for the second boundary value problem.

R. M. Evan-Iwanowski (Syracuse, N.Y.)

1730:

Slezinger, I. N. Castigliano's principle in nonlinear theory of elasticity. Akad. Nauk Ukrain. RSR. Prikl. Meh. **5** (1959), 38–44. (Ukrainian. Russian and English summaries)

The author formulates a chain of equivalent extremal problems obtained by a succession of transformations from the following: to find an extremum of the function

$$I[u_i;\lambda_{ij}]=\iiint_G\left[V(\lambda_{ij})-\sum_{i=1}^{3}u_iX_i\right]dx_1dx_2dx_3-\iint_{\Gamma_1}\sum_{i=1}^{3}u_iX_{n_i}d\omega$$

under suitable continuity and differentiability conditions, together with the equations $u_i=\bar{u}_i$ on a part Γ_2 of the surface Γ and $\lambda_{ij}=\partial u_i/\partial x_j$ inside G; here u_i are the displacement components in the medium, $\bar{u}_i$ are given displacements on Γ_2, V is the strain energy, $\bar{X}_i$ are the volume force components, and $\bar{X}_{n_i}$ are the surface force components given on the part Γ_1 of Γ. The last member of the chain represents the generalization of Castigliano's principle. To illustrate its use he solves the problem of the deformation of an elastic membrane with clamped edges.

R. N. Goss (San Diego, Calif.)

1731:

Teodorescu, P. P. A plane problem of the theory of elasticity with arbitrary body forces. Rev. Méc. Appl. 3 (1958), no. 1, 101–108. (Russian)

It is shown that in the case of plane stress with arbitrary body forces, the components of which are $X(x,y)$ and $Y(x,y)$, the stress components can be expressed in terms of a generalized Airy function F satisfying the equation

$$\Delta\Delta F=\int\frac{\partial^2X}{\partial y^2}dx+\int\frac{\partial^2Y}{\partial x^2}dy-\mu\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right),$$

where μ is Poisson's ratio. The decomposition of F into the sum of a biharmonic function $\bar{F}$ and a particular integral F' of the above equation leads to the conclusion that the first fundamental boundary-value problem of elasticity reduces for simply connected plane regions to the determination of the function $\bar{F}(x,y)$, for which on the contour are known either the values of $\bar{F}$ and its normal derivative or the values of the normal derivative and tangential derivative. Moreover $\bar{F}$ and its derivatives are one-valued functions in the region, provided the external load is in static equilibrium with the body forces.

R. N. Goss (San Diego, Calif.)

1732:

Sternberg, Eli; and Koiter, W. T. The wedge under a concentrated couple: a paradox in the two-dimensional theory of elasticity. J. Appl. Mech. 25 (1958), 575–581.

This paper was provoked by the following curious feature of the classical solution, due to Carothers, for the stresses in an infinite, two-dimensional wedge subjected to a concentrated couple at its vertex: as the opening wedge angle 2α approaches certain critical values (satisfying the equation $\tan 2\alpha=2\alpha$) the stresses given by the Carothers solution become infinite everywhere in the wedge. The authors observe that Carothers's results do not constitute a unique solution to the problem in its traditional formulation. They then consider a modified problem wherein the vertex moment is replaced by statically equivalent, bounded distributions of tractions over finite segments of the edges of the wedge in the neighborhood of the vertex; the limit (if it exists) of the solution to this modified problem as the loaded segment is contracted into the vertex is then presumed to provide the physically realistic solution to the original problem of Carothers. Results found by Mellin transform techniques confirm the validity of the Carothers solution for $2\alpha\leqq\pi$; for $\pi<2\alpha<2\alpha^*$ (where $2\alpha^*$ is the first of the critical angles) the Carothers solution is approached if the replacement loadings are restricted to be antisymmetric with respect to the wedge axis; and for $2\alpha\geqq2\alpha^*$ the solutions to the modified problem fail to approach a limit.

In a "note added in proof" the authors reject the limitation to an antisymmetrical replacement loading as unnecessarily restrictive, and their final conclusion is that the Carothers solution can be considered valid only for $2\alpha\leqq\pi$. This reviewer suggests that the purely antisymmetric replacement loading may not be devoid of physical significance, and that consequently the peculiar behavior of the Carothers solution as 2α approaches $2\alpha^*$ may merit further study.

B. Budiansky (Cambridge, Mass.)

1733:

Horvay, G.; and Mirabal, J. A. The end problem of cylinders. J. Appl. Mech. 25 (1958), 561–570.

The problem of a semi-infinite circular cylinder with free sides and given radially symmetric normal and radial tractions on the end is considered. This problem was solved by separation of variables by F. H. Murray, G. Horvay and H. Poritsky, and R. Trostel. The solution is a series of terms of the form $e^{-\gamma_k z}f_k(r)$. The γ_k are complex roots of a transcendental equation involving Bessel functions. They are very difficult to find. Moreover, the particular solutions obtained in this way give rise to both normal and radial end tractions.

The authors seek to construct some approximate solutions which give rise to either normal or shear tractions alone. The stresses are expressed in terms of two functions φ and Φ introduced by Sternberg and Sadowsky. The equilibrium conditions are satisfied. The functions φ and Φ are chosen to be products of functions of one variable, and the boundary conditions are imposed. The compatibility conditions are replaced by rendering the complementary energy stationary in a class of admissible φ and Φ.

In this way approximations to solutions with only normal traction or only radial tractions on the end of the cylinder are obtained. These solutions give rise to stresses of the form $e^{-\gamma z}f(r)$. The constants γ so obtained are quite close to γ_2, the γ_k with smallest real part. Since the solutions that are obtained satisfy end conditions entirely different from those obtained by means of the exact solution $e^{-\gamma_2 z}f_2(r)$, the closeness of γ to γ_2 is remarkable. It probably occurs because the term $e^{-\gamma_2 z}f_2(r)$ in the series expansion of the solution corresponding to most end conditions is dominant in the complementary energy integral.

If the end tractions depend upon two parameters, the exact solution for certain values of these parameters will not contain the term $e^{-\gamma_2 z}f_2(r)$. It will be dominated by the term $e^{-\gamma_4 z}f_4(r)$ where γ_4 is the γ_k with the second largest real part. By maximizing the real part of the exponential of the approximate solution with respect to various sets of parameters, the authors thus obtain good approximations to the values γ_4 and γ_6.

The stress distributions corresponding to the various approximate solutions are plotted.

H. F. Weinberger (College Park, Md.)

1734:

Amenzade, Y. A. Local tensions on twisting a round prismatic bar with elliptical non-coaxial aperture. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **5** (1959), 5–17. (Ukrainian. Russian and English summaries)

The author uses D. I. Šerman's method [Dokl. Akad. Nauk SSSR **63** (1948), 499–502; MR **10**, 651]. A numerical example is treated. *R. C. T. Smith* (Armidale)

1735:

Kuznecov, Yu. M. Effect of shear deformations on the magnitude of stresses from bending in continuous beams. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 1162–1166. (Ukrainian. Russian and English summaries)

The correction to the three moment equation necessary to allow for shear deformation is given. Some numerical results are presented in the form of graphs. *R. C. T. Smith* (Armidale)

1736:

Ramakanth, J. Finite torsion of aelotropic and composite cylinders. II. Bul. Inst. Politehn. Iaşi (N.S.) **4** (8) (1958), 75–84. (Russian and Romanian summaries)

By assuming a certain linear stress-strain relation the author treats the problem of finite torsion of a long circular cylinder composed of two different hexagonally aelotropic materials. The two materials are joined along a circular cylindrical surface having the same axis as that of the outer surface of the composite cylinder. The torsional couple is computed and evaluated for a Beryl-Apatite cylinder. [See also part I, viz.: J. Ramakanth, Z. Angew. Math. Mech. **35** (1955), 453–459; MR **17**, 685.] *L. E. Payne* (Newcastle-upon-Tyne)

1737:

Bassali, W. A.; and Nassif, M. Stresses and deflections in an elastically restrained circular plate under uniform normal loading over a segment. J. Appl. Mech. **26** (1959), 44–54.

"Within the limits of the small-deflection plate theory and using complex variable methods, an exact expression is developed in series form for the solution of the problem of a thin circular plate elastically restrained along the boundary and subjected to uniform normal loading over a segment of the plate. The elastic constraint considered includes as particular cases the rigidly clamped and simply supported boundaries. For a rigidly clamped boundary the results are expressed in finite terms. Some details of calculations of deflections, moments, and shears based on the theory are provided in tables and curves. Timoshenko's notation [*Theory of plates and shells*, McGraw-Hill, New York, 1940] is used in the paper." (Author's summary) *W. Schumann* (Zürich)

1738:

★Essenburg, F.; and Naghdi, P. M. On elastic plates of variable thickness. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 313–319. American Society of Mechanical Engineers, New York, 1958. xxvii + 864 pp. $20.00.

In the classical theory of thin elastic plates the influence of a possible variation of the thickness is taken into account by regarding the flexural rigidity in the basic equation for the normal deflection as a function of the coordinates in the plane of the plate. On the other hand this theory does not include the effect due to the inclination of the faces of the plate with respect to the middle surface, because these effects are in general assumed to be of small magnitude.

In this paper the authors intend to present a refined, but approximative, theory on the subject. It is based upon the following assumptions for the behaviour of certain quantities throughout the thickness: (1) linearity of the stresses and displacements parallel to the middle plane, (2) quadratic distribution of the shearing stresses, and (3) cubic distribution of the stresses normal to the plane of the plate. The corresponding expressions are arranged to satisfy the boundary conditions on the inclined faces of the plate and contain therefore, besides the stress resultants and couples, the derivatives of the thickness.

Then using Reissner's variational theorem a set of suitable global stress-displacement relations are established which, together with the equations of equilibrium, describe the problem. The theory is applied to the torsion of some special types of plates. *W. Schumann* (Zürich)

1739:

Zorski, Henryk. Some cases of bending of anisotropic plates. Arch. Mech. Stos. **11** (1959), 71–91. (Polish and Russian summaries)

Integral solutions are obtained and discussed for the bending of a semi-infinite anisotropic plate which has various conditions of loading and displacement along the edge. *L. S. D. Morley* (Farnborough)

1740:

★Csonka, P. Method for approximate calculation of shells curved in two directions. Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 49–58. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi + 362 pp.

Der Verfasser behandelt ein Näherungsverfahren zur Ermittlung des Membranspannungszustandes lotrecht belasteter Schalen. Er geht hierbei von der bekannten Pucherschen Spannungsfunktion aus. Für diese macht er einen mehrgliedrigen Ansatz mit freien Konstanten, der die Randbedingungen gliedweise erfüllt. Die Form der Mittelfläche bleibt innerhalb gewisser Grenzen ebenfalls offen, insofern ihre analytische Darstellung noch eine oder mehrere freie Konstanten enthält. Der Grundgedanke des Verfahrens besteht nun darin, diese freien Konstanten so zu bestimmen, daß die bekannte Pucher'sche Differentialgleichung 2. Ordnung für die Spannungsfunktion im Mittel so gut wie möglich erfüllt wird. Hierzu kann man etwa verlangen, daß das Integral über die Abweichungen bzw. über deren Quadrate verschwindet. Am vorteilhaftesten verwendet man jedoch die Methode von Galerkin.

Das Verfahren wird am Beispiel einer Kalottenschale über rechteckigem Grundriß erläutert. Eine nachträgliche Prüfung des Ergebnisses zeigt, daß die aus der angenäherten Spannungsfunktion berechnete vertikale Last überall sehr gut mit der vorgegebenen Belastung übereinstimmt; lediglich in den Ecken tritt eine gewisse

Abweichung auf, die aber 5% nicht überschreitet. Es zeigt sich außerdem, daß die endgültige Form der Schale, wie sie sich erst aus der Rechnung ergibt, durchaus annehmbar und vernünftig ist. Es scheint, daß gerade diese Anpassungsfähigkeit in der Wahl der Schalenform den guten Erfolg dieses Näherungsverfahrens bedingt.
W. Zerna (Hannover)

1741:

McIlroy, M. D. Linear deformations of conical shells. J. Aero/Space Sci. **26** (1959), 253–254.

Wie E. Reissner gezeigt hat, führt die lineare Theorie flacher Kegelschalen auf ein System zweier simultaner partieller Differentialgleichungen vierter Ordnung für eine Spannungsfunktion und die Normalverschiebung. Der Verfasser hat dieses System in komplexer Form zusammengefaßt und durch einen Reihenansatz gelöst. Jedes Glied verläuft in der Umfangsrichtung nach einer trigonometrischen Funktion mit *n* Wellen. Der Verlauf in radialer Richtung wird durch eine Funktion bestimmt, die einer Differentialgleichung vierter Ordnung genügt. Zwei Lösungen lassen sich durch Bessel-Funktionen darstellen und repräsentieren die Randstörungen, welche zum Schaleninneren abklingen: Die zwei übrigen Lösungen bestehen aus elementaren Funktionen und beschreiben Spannungszustände vom Typ der Membranlösung bzw. der dehnungslosen Biegung; hierin sind auch die Verschiebungen und Verdrehungen des starren Körpers enthalten. Zum Abschluß werden einige Rechenformeln für die Lösungsfunktionen angegeben. Der Artikel stellt eine kurze Zusammenfassung der Doktorarbeit des Verfassers dar, die dieser unter der Leitung von E. Reissner am Massachusetts Institute of Technology im Jahre 1958 verfaßt hat. *W. Zerna* (Hannover)

1742:

Valenta, Jaroslav. Die theoretische Lösung der dickwandigen geteilten Kreiszylinderschale. Apl. Mat. **3** (1958), 461–478. (Czech and Russian summaries)

Es handelt sich um die Ermittlung des Spannungszustandes in einem Körper, welcher entsteht, wenn man aus einem unendlich langen dickwandigen, kreiszylindrischen Rohr durch Schnitte längs der Erzeugenden ein Segment herausschneidet. Die Belastung verändert sich in Richtung der Zylinderachse nicht und hat in dieser Richtung keine Komponente; ferner wird sie symmetrisch in Bezug auf die Symmetrieebene des Segments vorausgesetzt. An den beiden freien Längsrändern werden die Spannungen bzw. Verschiebungen in der Querschnittsebene in symmetrischer Weise vorgeschrieben. Unter diesen Voraussetzungen liegt der Fall eines symmetrischen ebenen Formänderungszustandes vor. Nachdem bisher für dieses Problem nur eine Näherungslösung von Pöschl in expliziter Form vorliegt, gelingt es dem Verfasser durch Uebertragung einer von Hampl auf die dickwandige Kugelschale angewandten Methode eine Lösung zu finden, welche die Randbedingungen exakt zu erfüllen gestattet. Sie besteht aus einer Partikularlösung, welche den Einfluß der Belastung auf die äußere und innere Zylinderfläche berücksichtigt und einer homogenen Lösung, welche den Zustand infolge einer Belastung an den Längsrändern beschreibt.

Das behandelte Problem hat vor allem bei der Berechnung der Gehäuse von Dampfturbinen eine große, praktische Bedeutung. *W. Zerna* (Hannover)

1743:

Ambarcumyan, S. A.; and Peštmaldžyan, D. V. On the non-linear theory of sloping orthotropic shells. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 1, 15–26. (Russian. Armenian summary)

This paper is a continuation of the first author's previous investigations [Akad. Nauk Armyan. SSR. Dokl. **24** (1957), 153–159; Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **10** (1957), no. 2, 17–38; MR **19**, 1108, 1109]. The authors consider a sloping orthotropic shell (a sloping shell is an open shell of small curvatures) with distributed load normal to the middle surface, and assume that elements normal to the middle surface are non-deformable. When determining elongations and distortions due to shear the authors keep only those non-linear terms which result from normal displacements. They derive a system of differential equations which looks formidable for the general case but simplifies considerably for a spherical shell, for a circular plate, and for a rectangular shell of positive Gaussian curvature. The last case is solved by the authors when the load is constant and the dimensions and curvatures are given. The same case was previously solved by A. S. Vol'mir [*Gibkie plastinki i oboločki*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956; MR **19**, 788], who used an approximate but a more accurate procedure based on Kirchoff's hypothesis. Comparing the two solutions shows that the authors' solutions contain an error which in certain cases could reach 30%.
T. Leser (Aberdeen, Md.)

1744:

Argyris, J. H.; and Kelsey, S. The analysis of fuselages of arbitrary cross-section and taper. I. General theory. Aircraft Engrg. **31** (1959), 62–74, 101–112, 133–143.

This work presents a very efficient method for the structural analysis of stressed skin fuselage structures. The method of analysis is based on the "matrix force" method that was developed in 1955 by Professor Argyris. This analytical procedure permits a close integration of the analysis of the structure under consideration and the programming for a high-speed digital computer that is operating with a matrix interpretative scheme.

This important paper begins with a critical historical appraisal of previous work by many investigators of fuselage stress analysis. The "matrix force" method is then applied to the analysis of structures of diversified geometry that include most cases encountered in practice. The method allows for non-conical taper, double-cell cross-sections and structures involving doubly connected rings. The essential feature of the analysis is that it has been reduced to a highly standardized procedure that requires as input information only the simplest geometrical and elastic data. Another important feature is the development of a method of analysis that enables modifications of the structure to be incorporated without starting the process ab initio. The mathematical operations involved in the analysis are simple matrix algebraic processes. The most involved operation is that of matrix inversion.

In the opinion of the reviewer this paper is an important contribution to the theory of structures and should be read by all structural engineers concerned with stress analysis of fuselages. *L. A. Pipes* (Los Angeles, Calif.)

1745:

Csonka, P. Torsion of a square-shaped tube clasped in at both ends. Acta Tech. Acad. Sci. Hungar. **24** (1959), 379–390. (German, French and Russian summaries)

In the usual treatment of thin walled sections fixed at both ends only membrane-stresses are taken into consideration. The author relaxes the end conditions and treats a hollow square section by considering it as four plates which are fixed together only along their middle surfaces. He takes biharmonic forms for the displacements in plane and plate problems, and obtains the result that the bending and torsion resistance of the tube wall are sufficiently important to be taken into consideration. Similar results may be obtained by using the finite deflection theory. *B. R. Seth* (Kharagpur)

1746:

Janatka, Jaroslav. Determination of free torsional vibrations by means of difference equations. Apl. Mat. **1** (1956), 245–275. (Czech. Russian and English summaries)

The author considers a shaft of constant cross-section carrying identical discs at equal intervals. At each end of the shaft with identical discs is a flexibly joined shaft with discs of different moments of inertia. The shaft with identical discs the author calls "homogeneous", and the different discs joined flexibly at both ends are introduced through boundary conditions. The difference equations controlling torsional vibrations of such a system are presented by Karman and Biot in *Mathematical Methods in Engineering* [McGraw-Hill, New York, 1940; MR **2**, 77; pp. 454–460]. The author's contribution consists in the method of solution and in deriving a simple form of difference equation valid for an arbitrary number of discs at the boundaries. He arranges the frequency equations in such a way that one side depends on the number of discs in the homogeneous system while the other side depends on the dynamical spring constant replacing the systems which are joined at the boundaries. The transcendental frequency equation is solved graphically and the author explains his procedure as if he were completely ignorant of modern computing machinery. *T. Leser* (Aberdeen, Md.)

1747:

Fettis, Henry E. Note on the determination of higher modes of vibration by the Stodola or matrix-iteration method. J. Aero/Space Sci. **26** (1959), 317–318.

The paper describes a modification of the Stodola method by which difficulties with orthogonality conditions may be avoided in the computation of higher frequencies and modes. The author shows how to determine an iterating vector which is, in principle, orthogonal to the first $n-1$ modes; this vector is used to compute the nth mode. Unless the lower frequencies are known exactly, however, this iterating vector is not quite orthogonal to the first $n-1$ modes, and thus must be recomputed from time to time. *W. E. Boyce* (Troy, N.Y.)

1748:

Hieke, Max. Ein Beitrag zu den erzwungenen Schwingungen einer Membran. Z. Angew. Math. Mech. **38** (1958), 356–368. (English, French and Russian summaries)

The inhomogeneous differential equation for a circular membrane with fixed boundary subjected to a radially symmetric force is solved by a standard Fourier transform method, which is apparently new to the author. *H. F. Weinberger* (College Park, Md.)

1749:

Gerisch, Wolfgang. Zum Problem der an den Rändern fest eingespannten, elastischen Platte. Arch. Rational Mech. Anal. **2** (1958), 227–242.

Sets of functions $\{\phi_m\}$ satisfying "clamped edge" conditions $\phi_m = d\phi_m/dx = 0$ at the end points are useful for the approximate solution of problems in elasticity by the Rayleigh-Ritz and related methods. There is some advantage in the use of an orthonormal set and two such sets are given in this article.

The functions $\{\sin x \sin mx\}$, $m = 1, 2, \cdots$, discussed by M. M. Filonenko-Borodich [Akad. Nauk SSSR. Prikl. Mat. Meh. **10** (1946), 193–208; MR **7**, 437] when orthogonalized over the interval $(0, \pi)$ yield the set

$$\frac{\sin mx}{\sin x} - m \cos(m+1)x \quad (m = 1, 3, \cdots),$$

$$\sin(m+1)x \cot x - (m+1)\cos(m+1)x \quad (m = 2, 4, \cdots),$$

the normalizing factor in each case being

$$(2/\pi)^{1/2}\{m(m+1)\}^{-1/2}.$$

This set is shown to be complete in L_2.

The second set consists of associated Legendre polynomials

$$P_m{}^4(x) = (1-x^2)^2 \frac{d^4 P_m(x)}{dx^4}, \quad m \geq 4,$$

with the appropriate normalizing factor. Integrals over $(-1, 1)$ of products of derivatives of these polynomials reduce to very simple expressions. {This is probably the main advantage of this particular set of functions, though the orthogonality property does simplify expressions for the coefficients in the algebraic equations given by the Rayleigh-Ritz method. If desired, other functions with additional orthogonality properties can be obtained easily enough, e.g., if ϕ_m, ϕ_n are eigenfunctions of

$$\frac{d^4 y}{dx^4} + \lambda y = 0, \quad y(0) = \frac{d}{dx} y(0) = y(1) = \frac{d}{dx} y(1) = 0,$$

corresponding to different values of λ, then $\int_0^1 \phi_m \phi_n dx = \int_0^1 \phi_m'' \phi_n'' dx = \int_0^1 \phi_m^{\mathrm{IV}} \phi_n dx = 0$.}

The natural frequency and buckling load of an isotropic rectangular plate in shear are calculated using these functions. The results compare favourably with those of other authors and, in particular, are much more accurate than those of Iguchi [Ing.-Arch. **9** (1938), 1–12]. {The comparatively large error (5%) in Iguchi's result is however due to the use of a series method not equivalent to a variational method, as is shown by calculations of the reviewer [Austral. Counc. Aeronaut. Rep. **ACA-29** (1946); MR **9**, 122] using the Rayleigh-Ritz method with the same approximating functions as Iguchi.} *R. C. T. Smith* (Armidale)

1750:

Kaliski, Sylwester. The dynamic non-steady axially symmetric problem of a cylinder. Arch. Mech. Stos. **10** (1958), 793–810. (Polish and Russian summaries)

The paper presents an analysis of the subject problem assuming dynamic displacement functions to construct solutions for the vibration forcing field and the displacement tensor due to unit impulses. A system of integral equations is obtained and reduced to an infinite system of algebraic equations. A proof is given of the existence and uniqueness of the solution. *S. Levy* (Philadelphia, Pa.)

1751:

Volkov, A. N. Oscillations of a cylindrical shell in a flow of ideal fluid. Inž.-Stroitel. Inst. Kuibyšev. Sb. Trudov **27** (1957), 3–11. (Russian)

The effects on the transverse oscillations of a short, cylindrical shell by a flow of ideal fluid, without separation from the surface of the shell, are determined.

The increasing velocity of the flow increases the frequency of oscillations.

R. M. Evan-Iwanowski (Syracuse, N.Y.)

1752:

Foxwell, J. H.; and Franklin, R. E. The vibrations of a thin-walled stiffened cylinder in an acoustic field. Aero. Quart. **10** (1958), 47–64.

Plane harmonic sound waves are incident on an infinitely long stiffened cylinder normal to the cylinder's axis. Treated as a two-dimensional problem the scattered and internal sound fields are taken into account. It is shown that damping occurs due to radiation of acoustical energy from the cylinder, that the in vacuo resonance frequencies are shifted, and that many new resonance and antiresonance frequencies are introduced due to the coupling of cylinder vibration with wave motion inside the shell. *W. W. Soroka* (Berkeley, Calif.)

1753:

Mitra, M. The disturbance due to periodic surface traction in a semi-infinite medium of varying elasticity and density. Geofis. Pura Appl. **41** (1958), 86–90.

"In this paper, the shear displacement due to a transverse periodic surface traction on a semi-infinite elastic solid, in which the rigidity and density vary exponentially with the depth, has been obtained. Normal modes are calculated for high frequencies and the surface displacement obtained numerically in a particular case for a low frequency." (From the author's summary)

L. S. D. Morley (Farnborough)

1754:

Teodorescu, Petre P. A method for the stress solution of the plane problem of small damped motions. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 723–732. (Romanian. Russian and English summaries)

The author reviews briefly some of the works of previous investigators concerning small damped time-dependent motions in elastic media [cf. Clebsch, J. Reine Angew. Math. **61** (1863), 195–262; p. 195; Galitzin, Izv. Imper. Akad. Nauk (6) **6** (1912), 219–236].

The author solves the plane problem by means of only one stress-function, and expresses the different stress-strain relations in terms of this function for both condensation and distortion waves. Furthermore, the author shows that the solution of these dynamical problems can be expressed in terms of bi-metaharmonic functions, provided that a Fourier-series type of expansion can be allowed for the time element.

The coupled wave-equations (time independent) are, however, not explicitly solved in the present paper.

K. Bhagwandin (Oslo)

1755:

★Yeh, Gordon C. K.; and Martinek, Johann. Transient motion of a visco-elastic rectangular plate in fluid media. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 701–707. American Society of Mechanical Engineers, New York, 1958. xxvii + 864 pp. $20.00.

A thin rectangular plate is made of Maxwell-type visco-elastic material and separates two fluid compressible media, not necessarily with identical physical properties. The governing equation is derived in previous work by the authors [see Actes. Ninth Internat. Congr. Appl. Mech., Brussels, 1956, vol. 5, pp. 340–351, Univ. Bruxelles Press, 1957], and this is valid for small transverse motion of the plate. This paper considers the transient motion that ensues when the plate is released with zero velocity and acceleration from some prescribed arbitrary form of transverse deflexion. Attention is confined to the cases when all the edges of the plate are either simply-supported or clamped. It is shown that, at any time, the plate transverse deflexion is expressed as a linear combination of a set of space-dependent functions (essentially corresponding to those for the free vibrations of the analogous elastic plate) and a (single) set of time-dependent functions. The nature and degree of the damping (both acoustic and visco-elastic), which depends upon the relative values of certain physical constants, is reflected in the form assumed by these latter functions. The paper includes numerical data relating to the frequencies and modes of vibration for elastic plates. *H. G. Hopkins* (Fort Halstead)

1756:

Hilton, Harry H. On the representation of nonlinear creep by a linear viscoelastic model. J. Aero/Space Sci. **26** (1959), 311–312.

Baer [H. W. Baer, Proc. 2nd. U.S. Nat. Congr. Appl. Mech., pp. 569–576, Amer. Soc. Mech. Engrs., New York, 1954] has derived the following non-linear expression for creep strain in terms of stress:

$$\varepsilon = C_1\sigma^m + C_2\sigma^n t + C_3(1-\exp(-C_4 t)),$$

where the C are constants. It is pointed out that this behavior under constant temperature and stress can be represented by a Maxwell-Kelvin 4-parameter model, in which 2 parameters are functions of stress. The treatment can not be applied to experiments in which the stress varies with time. *B. Gross* (Rio de Janeiro)

1757:

Bland, D. R. On the foundations of linear isotropic visco-elasticity. Proc. Roy. Soc. London Ser. A **250** (1959), 524–549.

The microscopic structure of a linear viscoelastic material is represented by a finite network of linear elastic and viscous elements (springs and dashpots). The mech-

anics of this system is developed and expressions for the stored and dissipated energy and for the macroscopic equation of state are obtained. The general relation between stress and strain can be expressed by a total differential equation with constant coefficients involving derivatives of both stress and strain; this is known as the operator equation. Conventional model representations are discussed. The theory is generalized so as to include networks with an infinite number of parameters. Several explicit examples are given. *B. Gross* (Rio de Janeiro)

1758:

Charnes, A.; Lemke, C. E.; and Zienkiewicz, O. C. Virtual work, linear programming and plastic limit analysis. Proc. Roy. Soc. London Ser. A **251** (1959), 110–116.

The fundamental theorems of limit analysis [see, e.g., W. Prager, *An introduction to plasticity*, Addison-Wesley, Reading, Mass., 1959] provide lower and upper bounds for the load-carrying capacity of a rigid, perfectly plastic structure. It is well known that the problems of finding the greatest lower bound or the least upper bound can be formulated as linear programming problems, and that these bounds are identical. It was therefore suspected that each of the two linear programming problems is the dual of the other. In the present paper this is proved for plane frames under concentrated loads.

An important step in this proof is the derivation of a new parametric form of the equilibrium equations from the conditions of compatibility. The latter, which the authors seem to regard as new, are in fact well known to structural engineers. They are usually formulated in terms of "weights" that represent the angle changes at the test stations (points of application of loads and joints of the frame); these weights are normal to the plane of the frame and act at the test stations. Each loop formed by members of the frame that does not enclose other members furnishes three equations of compatibility, which may be interpreted as equations of equilibrium for the weights applied to the members of the loop. For the generic system of angle changes in the frame that satisfy the equations of compatibility, the principle of virtual work furnishes an equation of equilibrium containing the bending moments at the test stations and the given loads. Considered as a linear homogeneous equation in the angle changes, this general equation of equilibrium must be a consequence (i.e., a linear combination) of the equations of compatibility. This remark yields a parametric representation of all statically admissible systems of bending moments. Such parametric representations of statically admissible states of stress are of course widely used in the theory of structures, where the generic equilibrium state of stress is customarily represented by the superposition of an arbitrarily chosen state that is statically admissible under the given loads and a suitable number of states of self-stress. The new idea here is the derivation of a particular parametric representation from the equations of compatibility.

The authors announce their intention of investigating the computational advantages of this dual linear programming formulation in problems of limit design. Their results will be expected with interest because, in the past, linear programming formulations have not proved efficient in limit design. *W. Prager* (Providence, R.I.)

1759:

Hodge, P. G., Jr.; and Sankaranarayanan, R. On finite expansion of a hole in a thin infinite plate. Quart. Appl. Math. **16** (1958), 73–80.

A circular hole is expanded from zero radius in an initially uniform sheet of rigid/plastic material. The Tresca yield condition and potential are assumed, together with isotropic hardening in proportion to the energy dissipation. The solution is obtained in closed form, and is simpler than the Taylor-Hill solution for a non-hardening material with Tresca yield condition and Mises potential; however, the predicted thickness profiles differ radically. *R. Hill* (Nottingham)

1760:

Boyce, W. E. A note on strain hardening circular plates. J. Mech. Phys. Solids **7** (1959), 114–125.

A complete stress and displacement solution is given for a circular plate subjected to a uniform normal load and partially fixed at its outer radius. The plate material is assumed to be rigid for stresses less than the yield stress and to harden linearly in accord with Prager's hardening law. The degree of fixity is defined by k where $M=k\partial w/\partial r$ at the outer boundary. The solution is valid for all sufficiently small k. Graphical results show that the solution does not depend strongly upon k in the range considered.

The dependence of the solution upon the hardening coefficient and initial yield moment is also discussed.

P. G. Hodge, Jr. (Chicago, Ill.)

1761:

Piechocki, Wladyslaw. The state of stress in a circular disc due to the action of a source of heat. Rozprawy Inż. **6** (1958), 647–656. (Polish. Russian and English summaries)

The paper determines the state of stress generated in a circular isotropic disc by the action of a heat source placed at an arbitrary point within it. In the solution standard mathematical techniques are used.

J. Kestin (Providence, R.I.)

1762:

Nowacki, Witold. Thermal stresses due to the action of heat sources in a viscoelastic space. Arch. Mech. Stos. **11** (1959), 111–125. (Polish and Russian summaries)

A corresponding principle which links the linear theories of thermoelasticity and thermo-viscoelasticity [E. Sternberg, Proc. Third U.S. Nat. Congr. Appl. Mech., Providence, R.I., 1958, pp. 673–683, Amer. Soc. Mech. Engrs., New York, 1958; MR **20** #6852] is applied to the quasi-static determination of the thermal deformations and stresses arising from various instantaneous heat sources in a homogeneous and isotropic viscoelastic medium of infinite extent. The solutions obtained are in integral form and yield closed results in terms of elementary and error functions in the special instances of the Maxwell and the Kelvin solid.

E. Sternberg (Providence, R.I.)

STRUCTURE OF MATTER

See also 1343, 1826, 1856, 1857.

1763:

★**Seitz, Frederick; and Turnbull, David. (Editors) Solid state physics.** Advances in research and applications. Vol. 5. Academic Press Inc., New York, 1957. xv+455 pp. $11.00.

[For reviews of previous volumes, see MR **19**, 793, 794, 908, 1216.]

This volume will be equally of interest to theoretical and experimental physicists. In "Galvanomagnetic and thermomagnetic effects in metals", J.-P. Jan summarizes the work of the last 25 years on the many types of interaction of a magnetic field with electrical or thermal currents. Both the formal descriptive framework and the main experimental results are given.

"Luminescence in solids" by C. C. Kliek and J. H. Schulman is concerned largely with experimental work, grouped according to whether the luminescence involves absorption and emission in the same locality or energy transfer (with or without accompanying transfer of charge) to another locality before emission. Although some progress in the theory of luminescent processes is reported, it is clear that much is still obscure, especially where energy or charge transfer is involved.

In "Space groups and their representations", G. F. Koster begins with a brief account of the nature of space groups, including detailed listing of the irreducible representations of the point groups and description of the Bravais lattices. Then follows a thorough account of the irreducible representations of the space groups, illustrated in detail by a number of examples. Later sections deal with double groups, time reversal and a historical sketch.

"Shallow impurity states in silicon and germanium" by W. Kohn begins with a brief summary of properties giving information about the shallow impurity states (i.e., those with binding energies between 0.01 and 0.1 ev). Then he gives a detailed wave-mechanical discussion of the donor states and of refinements of the effective mass theory. The theory of acceptor states is dealt with more briefly, there having been less advance in this topic due to lack of experimental information and greater complication in the theory. In later sections, the effects of strain, static electric and magnetic fields and lattice vibrations are discussed briefly.

The volume concludes with an extensive review of "Quadrupole effects in nuclear magnetic resonance studies of solids" by M. H. Cohen and F. Reif, concentrating on those cases in which the nuclear electrical quadrupole interaction energy is small compared with the interaction energy of the nuclear magnetic moment with the internal field. The derivation of the quadrupole Hamiltonian is followed by a description of the spectra concerned. Then follow detailed discussions of the local electric field gradients (with special attention to ionic crystals), of "static" line broadening (with special development of a statistical approach), and of the role of quadrupole interactions in nuclear spin-lattice relaxation. The later sections deal with experimental studies in which the study of quadrupole effects is used in the elucidation of crystal structure, phase transformations and imperfections in crystalline solids. *M. S. Paterson* (Canberra)

1764:

★**Mott, N. F.; and Jones, H. The theory of the properties of metals and alloys.** Dover Publications, Inc., New York, 1958. xiv+326 pp. $1.85.

This is a reprint (with corrections) of the book published by Oxford University Press in 1936. Mott and Jones still stands as one of the best accounts of the basic principles of the wave-mechanical description of metallic properties, especially where alloys are concerned. The re-issue of this book is therefore very welcome after its unavailability for many years. *M. S. Paterson* (Canberra)

FLUID MECHANICS, ACOUSTICS

See also 1644, 1751, 1752, 1823, 1833, 1842, 1888, 1929.

1765:

Wu, C. S.; and Hayes, W. D. Crocco's vorticity law in a non-uniform material. Quart. Appl. Math. **16** (1958), 81–82.

This note extends Crocco's vorticity theorem [Z. Angew. Math. Mech. **17** (1937), 1–7] to non-uniform fluids with more than one constituent. Under the assumption of steady flow without viscosity in a conservative force field, and a thermodynamic equation of state expressed by

$$dh = Tds+\frac{dp}{\rho}+\sum\mu_k\, dn_k,$$

where μ_k and n_k are the molar chemical potential and specific molar concentration of the kth constituent, and p, T, h, s, and ρ are respectively the pressure, temperature, specific enthalpy, specific entropy, and density, the vorticity formula takes the form

$$q\times(\nabla\times q) = \nabla h_0 - T\nabla s - \sum\mu_k\,\nabla n_k.$$

In this equation q is the velocity vector and $h_0=h+\frac{1}{2}q^2+\Omega$ is the total enthalpy (Ω is the potential of the external force field). *D. Gilbarg* (Stanford, Calif.)

1766:

Drobot, S.; and Rybarski, A. A variational principle of hydromechanics. Arch. Rational Mech. Anal. **2** (1958/59), 393–410.

This paper is devoted to an intricate discussion regarding conditions under which hydrodynamical equations may be derived from variational principles, and, in particular, stresses the need for side conditions imposing conservation of mass. *M. J. Lighthill* (Manchester)

1767:

Nachbar, W.; Williams, F.; and Penner, S. S. The conservation equations for independent coexistent continua and for multicomponent reacting gas mixtures. Quart. Appl. Math. **17** (1959), 43–54.

This paper discusses the hydrodynamical equations for an n-component gas mixture, considered as a set of "coexistent continua", and shows the relation of these equations to the full kinetic-theory equations.

M. J. Lighthill (Manchester)

1768:

Dulov, V. G. Unsteady flow of gas from a cylindrical container. Vestnik Leningrad. Univ. **12** (1957), no. 13, 132–145. (Russian. English summary)

The author considers a cylindrical closed container with gas under high pressure, when suddenly an orifice at the center of one of the bases is being opened and the gas flows out of the container. The problem consists of investigating during the flow the variations of gas parameters inside the container. The author assumes a model of one-dimensional adiabatic unsteady flow of an ideal gas and proceeds almost step by step as K. P. Stanyukovich, *Neustanovivšiesya dviženiya splošnoĭ sredy* [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955; MR **18**, 440; pp. 154–173]. The difference between Stanyukovich's case and the author's case is that in the former the whole base of the cylindrical container is suddenly removed, whereas in the latter only an orifice is being opened; thus the author has to deal with reflections of rarefaction waves from the front wall as well. *T. Leser* (Aberdeen, Md.)

1769:

Patraulea, N. N. Deux généralisations du problème de l'aile au volet fluide. Rev. Méc. Appl. **2** (1957), no. 1, 29–38.

The author generalizes his theory of the jet flap in an incompressible fluid flow to obtain more precise results. The first generalization consists of a superposition of a uniform flow field and an induced field due to singularities, imitating the additional velocity components around a jet flap. In the complex plane (x, iy) the airfoil is represented by a straight-line segment. Moreover, the wake up to infinity is taken into account. By a conformal mapping this is transformed into a semi-circle and it is assumed that the complex velocity potential is composed of three parts (due to circulation, perturbation, etc.). The physical boundary conditions are sufficient to find the final form of the complex velocity potential. The use of complex variable technique restricts this method to a two-dimensional flow. In the second step the author generalizes this technique to a finite airfoil in the following way: the circulation along the span of a finite airfoil is assumed to be elliptical. This, in turn, allows one to assume that the induced velocities are practically independent of the co-ordinate along the span (classical result). Hence, it is sufficient to solve a two-dimensional problem in the plane of symmetry and the problem reduces to the previous one. Because of the influence of the induced velocities due to the wake the obtained equation is non-linear and must be solved by an iteration process. The knowledge of the velocity pattern allows one to find the pressure distribution. *M. Z. v. Krzywoblocki* (Urbana, Ill.)

1770:

Polášek, Jan. Theory of thin airfoil sections in non-uniform flow. Apl. Mat. **1** (1956), 44–58. (Czech)

The author develops a general theory of a thin airfoil section in a two-dimensional non-uniform flow of a non-compressible fluid. The equation representing the shape of the airfoil section is given in a closed form and also as a Fourier series. The author solves the direct problem of finding the velocity distribution when the shape of the airfoil is given, as well as the inverse problem when the velocity distribution is given and the shape of the airfoil section is to be determined. *T. Leser* (Aberdeen, Md.)

1771:

Polášek, Jan. Theory of thin airfoil sections in non-uniform flow. Apl. Mat. **1** (1956), 119–135. (Czech. Russian and English summaries)

This paper is the conclusion of the paper reviewed above. The author gives a chapter on numerical computation of the results and four appendices, three of which have derivations omitted in the main text and one of which is on solving a system of linear equations. *T. Leser* (Aberdeen, Md.)

1772:

Sretenskiĭ, L. N. The Cauchy-Poisson problem for the surface of separation of two flowing fluids. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1955**, 505–513. (Russian)

The author considers the following generalization of the Cauchy-Poisson problem. Let a fluid of density ρ_1 [ρ_2] fill the region $y<0$ [$y>0$] and move to the left with velocity c_1 [c_2]. At time $t=0$ let the interface $y=0$ be suddenly disturbed. How does the fluid behave for $t>0$? Making the usual assumption of small disturbance, the author derives formulas for the velocity potentials in the two fluids and for the form of the interface. As he points out, they are not applicable to all choices of initial disturbance. An allowable one is studied in some detail. Various interesting differences from the usual behavior (i.e., when $\rho_2=0$) occur. *J. V. Wehausen* (Berkeley, Calif.)

1773:

De, S. C. Contributions to the theory of Stokes waves. Proc. Cambridge Philos. Soc. **51** (1955), 713–736.

In a note written for his collected *Mathematical and physical papers* [vol. 1, University Press, Cambridge, 1880; pp. 314–326], Stokes presented a method for computation of periodic waves of permanent type and of finite amplitude and carried through to the third order the computation of the coefficients in an associated series for fluid of finite depth. The author finds the coefficients to the fifth order. On the basis of the derived formulas, tables of c^2m/g (c the wave velocity and m the wave number) and of certain other quantities were prepared and are included. Comparison is made between predictions of this approximation and of the cnoidal-wave approximation. *J. V. Wehausen* (Berkeley, Calif.)

1774:

Haque, S. M. A. The effect of eddy viscosity on the velocity profile of steady flow in a uniform rough channel. J. Fluid Mech. **5** (1959), 310–316.

This is a hydraulic theory for steady flow under gravity down a uniform slope: the velocity is assumed to vary only in the horizontal direction normal to the direction of the mean flow. Bottom friction is included through a force proportional to the square of the velocity, and lateral mixing through an eddy viscosity term. The velocity is worked out for rectangular, triangular and trapezoidal cross-sections, with the eddy viscosity constant or varying with velocity and position. Numerical values are given to indicate the influence of eddy viscosity on the lateral distribution of mean velocity.

F. Ursell (Cambridge, England)

1775:

Tanaka, Kiyoshi. Theory of the surges in the open channels. Tech. Rep. Osaka Univ. 8 (1958), 299–311.

The equations of long-wave theory with a quadratic resistance term are used, and small perturbations about steady flow are treated. The linearized equations are of hyperbolic type and are solved by Riemann's method. The author treats the initial-wave problem for an infinite channel, and the forced unsteady motion when the amplitude is prescribed at one point of an infinite channel.

F. Ursell (Cambridge, England)

1776:

Eckart, Carl. Surface wake of submerged sphere. Phys. Fluids **1** (1958), 457–461.

The wake of the title is the system of Kelvin ship waves generated by the sphere, and the boundary condition at the free surface is linearized. The author is evidently unaware of the extensive literature on this problem [e.g. T. H. Havelock, Proc. Roy. Soc. London, Ser. A **93** (1917), 526–532; **95** (1918), 358–365; **121** (1928), 517–523; **131** (1931), 275–285. Also G. Green, Philos. Mag. (6) **36** (1918), 48–63; and H. Lamb, Proc. Roy. Soc. London, Ser. A **111** (1926), 14–25; *Hydrodynamics*, 6th ed., University Press, Cambridge, 1932; § 256a].

F. Ursell (Cambridge, England)

1777:

★Langlois, W. E. Creeping viscous flow through a two-dimensional channel of varying gap. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 777–783. American Society of Mechanical Engineers, New York, 1958. xxvii + 864 pp. $20.00.

The slow steady two-dimensional motion of incompressible viscous liquid through a channel of varying width is considered by three different approximate methods, and the results of the methods are compared.

W. R. Dean (London)

1778:

Ballabh, Ram. On the possibility of steady Beltrami flow in a viscous liquid. Math. Student **26** (1958), 21–24.

The author shows that steady, viscous, incompressible, Beltrami flows do not exist in three cases. First, the author considers an unbounded fluid with constant velocity at infinity. By use of Green's theorem in three dimensions, it is shown that the vorticity (and hence the velocity) must vanish. Secondly, the case of a bounded fluid with fixed boundaries is discussed. Finally, the author considers the case when the streamlines form a congruence of curves normal to a special family of equi-potential surfaces.

N. Coburn (Ann Arbor, Mich.)

1779:

Orovyanu, T.; and Ionesku, P. Determination of fluid leakage through annular clearance between piston and cylinder in a pump. Rev. Méc. Appl. **2** (1957), no. 2, 69–78. (Russian)

The paper deals with the case of axial laminar flow of an incompressible, viscous fluid in an annular gap between two concentric and eccentric infinite cylinders. The problem is solved by standard mathematical techniques and its importance is conditioned by the intended application.

J. Kestin (Providence, R.I.)

1780:

Heyda, James F. A Green's function solution for the case of laminar incompressible flow between non-concentric circular cylinders. J. Franklin Inst. **267** (1959), 25–34.

The rectilinear flow under pressure of a viscous liquid in a straight-pipe bounded by non-concentric circular cylinders is found using the appropriate Green's function. Cf. the distribution of stress in the torsion of the corresponding cylinder [Love, *Theory of elasticity*, University Press, Cambridge, 1892, 1920, 1934; chap. 14].

W. R. Dean (London)

1781:

Săvulescu, Şt. Sur une détermination des caractéristiques de la couche limite par l'utilisation de certains profils type de vitesses et de températures. Rev. Méc. Appl. **2** (1957), no. 1, 39–53.

An approximate method is developed for determining the skin friction in boundary layers including the effects of turbulence compressibility and suction. The momentum and energy integrals are used together with a velocity profile expressed in terms of the stream function. Not surprisingly perhaps, the results seem to be about as accurate as those obtained using the Kármán-Pohlhausen method.

K. Stewartson (Durham)

1782:

Rozin, L. A. The growth of a laminar boundary layer on a flat plate set impulsively into motion. J. Appl. Math. Mech. **22** (1958), 568–575 (407–412 Prikl. Mat. Meh.).

The author considers the problem of a semi-infinite flat plate that is suddenly caused to move with a velocity U_0 in a direction parallel to itself and normal to its leading edge. The Navier-Stokes equations reduce to a single non-linear equation for the stream function ψ in this case, and a solution of this is obtained in the form of a series $\psi=\sum_0^\infty \psi_n$, where $\psi_0=U_0y$, and the functions ψ_n for $n>0$ satisfy linear differential equations. It is shown that this series is equivalent to a power series in the time t. The solution is carried as far as the second term ψ_1, with the understanding that $U_0^2t/\nu \ll 1$. Analytical and numerical results are obtained for the velocity ratio u/U_0 as a function of $\xi=x/(\nu t)^{1/2}$ and $\eta=y/(\nu t)^{1/2}$, where x is the distance from the leading edge and y is the normal distance to the surface. The influence of the leading edge is found to be confined to distances of the order of $(\nu t)^{1/2}$ upstream and downstream of the leading edge. For $\xi>2$, the solution rapidly approaches the solution for a plate infinite in both directions.

D. W. Dunn (Ottawa, Ont.)

1783:

Rozin, L. A. An approximate method of integration of the non-stationary laminar boundary layer equations for an incompressible fluid. Prikl. Mat. Meh. **21** (1957), 615–623. (Russian)

Let a one-parameter family of u-profiles be given in the form $u/U=\phi'(\eta,\beta)$, $\eta=yA(\beta)/\delta^*$, where β and the displacement thickness δ^* depend on x and t, and $A(\beta)=\int_0^\infty (1-\phi')d\eta$. The quantity $f\equiv -A^2(\beta)\phi''(0,\beta)$, which equals $(\partial U/\partial x+U^{-1}\partial U/\partial t)\delta^{*2}/\nu$ if u is to satisfy the boundary layer equation of motion at $y=0$, may be taken as the form parameter instead of β; and the two coefficient functions $h\equiv$ (momentum thickness)$/\delta^*$ and $\zeta\equiv$

$A(\beta)\phi''(0, \beta)$, which both appear in the momentum-integral equation, may be considered as functions of f. The author notes that for certain profiles such as Hartree's and Watson's [the latter defined by a linear third order equation in E. J. Watson, Proc. Roy. Soc. London Ser. A **231** (1955), 104–116; MR **17**, 99; and denoted by ϕ_0] the functions $h(f)$ and $\zeta(f)$ are very nearly linear; h in particular may be approximated by a constant. In this approximation the momentum-integral equation becomes a linear partial differential equation for δ^{*2}/ν; its solution determines $f(x, t)$ and thus $\beta(x, t)$ and may be improved by iteration. Examples include the separation at the near stagnation point of a circular cylinder moving according to U_0t^n and the small oscillations of a semi-infinite plate in a steady stream U_0.
G. Kuerti (Cleveland, Ohio)

1784:

Becker, Ernst. Berechnung von Reibungsschichten mit schwacher Sekundärströmung nach dem Impulsverfahren. Z. Flugwiss. **7** (1959), 163–175. (English and French summaries)

Based on the momentum theorem in the form given by Gruschwitz, a simple method of approximation is derived for estimating the development of laminar or turbulent boundary layer with a weak secondary flow, after applying a correction, invariably neglected when writing down these momentum equations in the form given by Gruschwitz. The well-known method of computation due to Holstein and Bohlen for laminar flow is contained, as a special case, in the formulae reported here.
Author's summary

1785:

Imai, Isao. On the heat transfer to constant-property laminar boundary layer with power-function free-stream velocity and wall-temperature distributions. Quart. Appl. Math. **16** (1958), 33–45.

The author treats the problem described in his title by using an improved version of the WKB method to solve the ordinary differential equation for the temperature distribution. The approximation involved is basically one of "large" Prandtl number, but enough terms are taken so that the Prandtl number would have to be quite small to cause any substantial error. The method is compared, favourably, with a number of others which have been used.
M. J. Lighthill (Manchester)

1786:

Novikov, E. A. Concerning a turbulent diffusion in a stream with a transverse gradient of velocity. J. Appl. Math. Mech. **22** (1958), 576–579 (412–414 Prikl. Mat. Meh.).

The diffusion of an impurity from an instantaneous point source in a turbulent stream with a uniform gradient of mean velocity is considered. By a change of variables, the diffusion equation becomes

$$\frac{\partial q}{\partial t} = S_{\mu\nu}(t)\frac{\partial^2 q}{\partial y_\mu \partial y_\nu},$$

where $S_{\mu\nu}(t)$ is a generalized eddy diffusivity, and the general solution is

$$q = \frac{Q}{(4\pi)^{3/2}\sqrt{(\mathrm{Det}(\tau))}}\exp\{-\tfrac{1}{4}T_{\mu\nu}^{-1}y_\mu y_\nu\},$$

where $T_{\mu\nu}^{-1}$ is the matrix inverse to $T_{\mu\nu}(t) = \int_0^t S_{\mu\nu}(t')dt'$. For stationary turbulence, it is found that the apparent diffusivity in the stream direction exceeds the true eddy diffusivity by an amount that increases as the square of the time of diffusion. It is correctly stated that the interaction between the velocity gradient and the diffusion produces this effect and is responsible for the frequent observation that clouds of diffusing material are elongated in the wind direction. This has already been pointed out by G. I. Taylor [Proc. Roy. Soc. London Ser. A **223** (1954), 446–468]. *A. A. Townsend* (Cambridge, England)

1787:

Pogodin, Iu. Ia.; Suchkov, V. R.; and Ianenko, N. N. On travelling waves of gas dynamic equations. J. Appl. Math. Mech. **22** (1958), 256–267 (188–196 Prikl. Mat. Meh.).

The problem considered is that of finding solutions of the set of linear equations

$$a_{ijk}(u_1, \cdots, u_m)\frac{\partial u_j}{\partial x_k} = 0 \quad (i, j, k = 1, \cdots, m)$$

such that the functional relationship

$$\varphi(u_1, \cdots, u_m) = 0$$

is satisfied. A formal solution method is given and then applied to a particular case—the two-dimensional unsteady motion of a polytropic gas. The method leads to a quasi-linear partial differential equation with one less independent variable than the original set. When the flow is isothermal this equation becomes linear and readily soluble and a specific example is worked out for this special case.
J. J. Mahony (Sydney)

1788:

Chu, Boa-Teh; and Kovásznay, Leslie S. G. Non-linear interactions in a viscous heat-conducting compressible gas. J. Fluid Mech. **3** (1958), 494–514.

The non-linear equations of a heat-conducting viscous compressible gas flow are analyzed by means of an expansion in terms of a small amplitude parameter α and through the use of a second small parameter ε which measures the relative influences of inertia and viscosity and heat conduction. The three modes of fluctuation that appear in an uncoupled fashion (providing the flow is far from boundaries) in the first order or linearized theory are carried over to discuss the higher order non-linear interactions between modes. Mathematically this is done by writing apparent mass-addition, body force, and heat addition contributions as non-homogeneous terms in the set of partial differential equations. These are then computed at each order of approximation in α in terms of lower order quantities and provide the interaction information.

The fluctuation modes are labeled sound, entropy and viscosity, corresponding to the way they separate in the first order theory. The eighteen possible bilateral interactions of these modes associated with the apparent body force, mass, and heat addition terms of the second order theory are studied for the special case in which all three fluctuations have amplitude and wave length scales of the same order of magnitude. For the case in which ε is also small the interactions are set out in full. These show,

for example, that the sound mode and the entropy mode interact to produce all three basic modes. On the other hand, the sound mode's interaction with itself produces only a sound contribution in the form of wave front steepening or self-scattering but no vorticity or entropy effects, to this order of magnitude. The other interactions are similarly described.

This seems to be an interesting and useful classification of non-linear effects and furthermore a complete one. It has been apparently motivated by interest in turbulence measurements. It is pointed out that the forms of modal coupling described appear when the flow is not near walls or fluid boundaries. At boundaries, conditions are usually to be satisfied in terms of fluid velocities, stresses or temperatures and not in terms of the modal separation employed. It would be of interest to compare the order of magnitude of non-linear couplings produced at boundaries with those delineated in this paper.

Hirsh Cohen (Yorktown Heights, N.Y.)

1789:

Grigorian, S. S. Limiting self-similar, one-dimensional, non-steady motions of a gas (Cauchy's problem and the piston problem). J. Appl. Math. Mech. **22** (1958), 417–430 (301–310 Prikl. Mat. Meh.).

An interesting class of particular solutions of the self-similar type is studied for which the initial density, entropy and gas velocity depend exponentially on the distance. The discussion of the integral curves of the associated ordinary differential equations indicates that ranges of the disposable parameters exist in which either the similar solution cannot be extended indefinitely in time, even when a shock is admitted, or an infinite number of solutions satisfies the same initial conditions. The paper also gives a brief sketch of a self-similar solution for the gas motion between a very strong shock and a spherical, cylindrical or flat body expanding exponentially in time.

R. E. Meyer (Providence, R.I.)

1790:

Diaz, J. B. Two-dimensional flow at high subsonic speeds past wedges in channels with parallel walls. J. Math. Phys. **38** (1959/60), 75–76.

The author shows that fundamental solutions given by Cole [same J. **30** (1951), 79–93; MR **15**, 263] and Helliwell [J. Fluid Mech. **3** (1958), 385–403; MR **19**, 1006] were given previously by Weinstein [Trans. Amer. Math. Soc. **63** (1948), 342–354; Naval Ordnance Lab. Rep. **NOLR-1132** (1950), 73–82; MR **9**, 584; **12**, 875].

H. C. Levey (Nedlands)

1791:

Landahl, Mårten, T. Theoretical studies of unsteady transonic flow. IV. The oscillating rectangular wing with control surface. Flygtekn. Försöksanstalt. Rep. no. 80 (1959), 28 pp.

Continuant une série de trois rapports précédents [mêmes Rep. no. 77, 78, 79 (1958); MR **21** #535, #536, #537], l'auteur étudie l'aile rectangulaire en utilisant toujours la transformation de Fourier pour se ramener ici au problème de la diffraction par une fente. La solution de ce problème est obtenue, sous forme de séries convergentes pour toute fréquence réduite non nulle, grâce à un procédé alterné de Schwarzschild. Au départ on respecte seulement les conditions sur la fente, c'est à dire sur l'aile, puis, en utilisant la solution de Sommerfeld relative à la fente semi-infinie, on détruit alternativement les résidus de portance à l'extérieur de l'aile. En revenant aux variables physiques, l'auteur donne des formules qui résolvent en principe le problème pour une aile rectangulaire en déformation arbitraire. En passant aux applications, l'auteur se limite à la considération de déformations indépendantes de l'envergure et ne conserve que trois termes dans ses séries. La convergence du procédé alterné parait être très rapide. L'interprétation des termes successifs est la suivante. Premier terme: théorie des tranches; deuxième terme: juxtaposition de deux ailes semi-infinies sans interaction mutuelle des bords; troisième terme: correction d'interaction mutuelle des bords. La solution à deux termes a déja été donnée dans Rep. 78 [loc. cit.].

Des résultats numériques sont présentés pour la portance et le moment dus au battement et au tangage, ils montrent que les diverses méthodes d'approche se recoupent. Soient k la fréquence réduite (basée sur la corde) et A l'allongement (rapport envergure-corde), les domaines de validité des diverses méthodes sont respectivement: 1 terme: tranches $A\sqrt{(k+k^2/2)}>8.5$; 2 termes: Rep. 78, $A\sqrt{(k+k^2/2)}>1.3$; 3 termes: actuel, $A\sqrt{(k+k^2/2)}>0.8$; faible allongement: Rep. 79, $A\sqrt{(k+k^2/2)}<1.0$. L'inefficience de la méthode des tranches est ainsi nettement mise en évidence.

La stabilité d'un aileron rectangulaire est étudiée, confirmant que les effets tridimensionnels agissent fortement dans le sens de la stabilité. Ainsi, un aileron noyé dans l'aile est stable pour toutes fréquences réduites si son allongement est inférieur à 3.5, alors que pour un aileron en bout d'aile il suffit de descendre en dessous de 4.5.

J. P. Guiraud (Paris)

1792:

Landahl, Mårten T. Theoretical studies of unsteady transonic flow. V. Solution for the delta wing and wings of general polygonal planforms. Flygtekn. Försöksanstalt. Rep. no. 81 (1959), 22 pp.

Ceci est la dernière partie d'une série de 5 rapports consacrés aux écoulements transsoniques non stationnaires. Les rapports précédents étaient essentiellement consacrés à l'aile rectangulaire. Une transformation laissant invariante l'équation du potentiel des vitesses, mais changeant une forme en plan triangulaire en une forme en plan rectangulaire permet d'obtenir la solution complète du problème relatif à l'aile delta. Par composition il est possible ensuite de passer à l'aile polygonale. La mise en œuvre numérique de la méthode parait necessiter d'énormes moyens.

J. P. Guiraud (Paris)

1793:

Ehlers, F. Edward. The method of characteristics for isoenergetic supersonic flows adapted to high-speed digital computers. J. Soc. Indust. Appl. Math. **7** (1959), 85–100.

The author considers the characteristic form of the equations for steady supersonic plane (axisymmetric) flow relative to rectangular (cylindrical) coordinates. The dependent variables are coordinates x, y; the slope ζ of velocity with respect to the x-axis; $\beta=(M^2-1)^{1/2}$; and entropy s. The differential equations, boundary

conditions, free boundary conditions, and conditions at a shock following uniform flow are converted to first order difference equations. To organize a machine calculation, methods are developed to determine from given (or previously calculated) values of x, y, ζ, β, and s at two points of a characteristic net approximate values of these functions at an intersection of (1) two Mach lines; a Mach line and (2) a rigid boundary; (3) the axis of symmetry; (4) a free boundary; (5) a shock following uniform flow. The values so found are used to improve estimates of coefficients in the difference equations, thus providing the basis of an iterative method. An application to calculation of flexible throat wind tunnel nozzles is sketched. No attempt is made to estimate truncation errors.

J. H. Giese (Aberdeen, Md.)

1794:

Coburn, N. Simple waves in the steady rotational plane supersonic flow of a polytropic gas of constant entropy. Michigan Math. J. **5** (1958), 129–137.

Les résultats obtenus dans ce travail sont obtenus comme applications des équations générales de la théorie des caractéristiques antérieurement données par l'auteur [Quart. Appl. Math. **15** (1957), 237–248; MR **19**, 1006]. Il s'agit d'écoulements plans stationnaires à entropie constante mais à enthalpie génératrice variable et par suite d'écoulements rotationnels. L'auteur étudie la possibilité d'écoulements par ondes simples (bicaractéristiques rectilignes). Il met en évidence l'existence de solutions pour lesquelles ces droites sont concourrantes ou tangentes à un même cercle. Lorsque $\gamma=3$, on peut montrer que ce sont les seuls écoulements par ondes simples possibles. Lorsque l'écoulement est constamment sonique, on est conduit à un résultat curieux: les seuls écoulements possibles sont des écoulements à ondes centrées, les lignes de courant étant des cercles concentriques.

P. Germain (Paris)

1795:

Bulakh, B. M. Remarks on the paper by L. R. Fowell, "Exact and approximate solutions for the supersonic delta wing". J. Appl. Math. Mech. **22** (1958), 562–567 (404–407 Prikl. Mat. Meh.).

The author criticizes that part of the paper cited in his title which deals with flow over the expansion surface of a flat delta wing at incidence. He objects to the curve adopted by Fowell as the boundary of the region of Prandtl-Meyer expansion in the plane of the cone-field variables. This curve should have been a characteristic of the equations of motion, but instead was taken as the parabolic curve of those equations, which the author points out (as it appears to the reviewer, rightly) is not the same. The difference is used by the author to explain discrepancies between the computational and the experimental results given in Fowell's paper.

M. J. Lighthill (Manchester)

1796:

Guiraud, Jean-Pierre. Approximations dans les écoulements hypersoniques tridimensionnels de fluide parfait. C. R. Acad. Sci. Paris **248** (1959), 2443–2445.

Several results of Newtonian theory are exhibited without proof. Expressions are given for streamline curvature and pressure in the shock layer to the second approximation; that is, neglecting squares and products of $\gamma-1$ and M^{-2}. For an inclined circular cone the surface pressure and streamline shape are shown neglecting also squares of angle of attack. The normal force on any slightly inclined body of revolution is given to the first approximation of Newtonian theory.

M. D. Van Dyke (Stanford, Calif.)

1797:

Belotserkovskii, O. M. Flow with a detached shock wave about a symmetrical profile. J. Appl. Math. Mech. **22** (1958), 279–296 (206–219 Prikl. Mat. Meh.).

The paper describes an important application of the new technique developed by Dorodnitsin [Trudy 2-go. Vses. Mat. S"ezda 2 M. Akad. Nauk SSSR 1956, 78] for solving non-linear second order partial differential equations of mixed type. In the present problem, concerning supersonic flow past blunt nosed bodies, a detached shock wave is formed ahead of the body. The flow behind this is subsonic near the stagnation point but becomes supersonic further downstream. The supersonic region can be divided into two parts, one which influences conditions in the subsonic region and one which does not. The flow in the subsonic region and the first part of the supersonic region must be considered as a whole and the problem of determining this is formulated mathematically as a non-linear analogue of Tricomi's problem. To solve this approximately the author introduces a coordinate ξ normal to the body which has value zero on the body and value unity at the unknown shock. The angle θ between this normal and the axis of symmetry is the second coordinate. The region between the body and the shock is divided by lines of constant ξ into equal strips, the number of strips depending on the order of approximation. In the first approximation dependent variables occurring in the equations of motion are assumed to vary linearly with ξ. In the second approximation they vary quadratically, and so on. The equations of motion are then reduced to sets of ordinary equations in θ which are integrated numerically. The boundary conditions are determined by symmetry properties at $\theta=0$ and by the condition that the solution must pass smoothly through singularities which arise near the sonic line. The non-singular solutions are unique. It appears that the second approximation is adequate for most purposes, giving excellent agreement with experimental results.

M. Holt (Providence, R.I.)

1798:

Belocerkovskiĭ, O. M. Computation of flow around a circular cylinder with detached shock wave. Vyčisl. Mat. **3** (1958), 149–185. (Russian)

This is a more detailed account of the author's already well-known numerical procedure for treating supersonic flow past blunt bodies [#1797 above]. The results for circular cylinders at Mach number 2.13, 2.5, 3, 4, and 5 are tabulated extensively, and compared with approximate theories and with experiments (including Russian ones).

M. D. Van Dyke (Paris)

1799:

Whitham, G. B. A new approach to problems of shock dynamics. II. Three-dimensional problems. J. Fluid Mech. **5** (1959), 369–386.

The author's hypothesis [same J. **2** (1957), 145–171; MR **19**, 206] regarding the motion of a shock wave into still air is that, if lines which are orthogonal to each successive position of the shock wave are called rays, then the cross-sectional area A of a tube of rays varies along the tube in proportion to a universal function of M, the Mach number of the shock wave. To make use of this result in three-dimensional problems, the author writes the equation of the shock wave as

$$a_0 t = \alpha(x, y, z),$$

where a_0 is the velocity of sound in the undisturbed fluid and t is the time. Then

$$M = \frac{1}{|\nabla\alpha|}, \quad \nabla\cdot\left(\frac{M}{A}\nabla\alpha\right) = 0,$$

with A a known function of M.

In the important case of diffraction of an identically plane shock of Mach number M_0 by a solid obstacle, $\partial\alpha/\partial n = 0$ on the obstacle and $\alpha \sim x/M_0$ at infinity. Therefore, α is the same as the velocity potential ϕ of a compressible hypothetical-gas flow past the body, which would satisfy the same boundary conditions, and the same equation $\nabla\cdot(\rho\nabla\phi) = 0$ with ρ some function of $|\nabla\phi|$. The equipotentials of the supersonic flow problems are the shock positions of the diffraction problem.

This analogy method is ingeniously pursued to get a number of general results, and also detailed information on shock wave diffraction by cones and slender bodies.

M. J. Lighthill (Manchester)

1800:

Tan, H. S. Nose drag in free-molecule flow and its minimization. J. Aero/Space Sci. **26** (1959), 360–366.

The drag of a body in a uniform stream of gas of velocity V and temperature T is obtained on the assumption that all molecules colliding with the body surface are adsorbed and afterwards emitted with a Maxwellian velocity distribution corresponding to the temperature T_r of the surface. No collisions with other molecules occur. The drag coefficient depends on T_r/T and on the "speed ratio" $S = V(2RT)^{-1/2}$, where R is the gas constant per unit mass. The functional form becomes simpler as $S \to 0$ or as $S \to \infty$, and the author studies the problem of minimizing the drag coefficient in these two limiting cases by optimum selection of nose shape. *M. J. Lighthill* (Manchester)

1801:

Redwood, M. Velocity and attenuation of a narrow-band, high-frequency compressional pulse in a solid wave guide. J. Acoust. Soc. Amer. **31** (1959), 442–448.

The Pochhammer-Chree solution does not correctly describe the propagation of narrow-band high-frequency pulses in solid waveguides, but the description becomes correct when a slightly different form of the solution is chosen. *C. H. Papas* (Pasadena, Calif.)

1802:

Srivastava, A. C. Rotation of a plane lamina in non-Newtonian fluids. With an appendix by S. D. Nigam: Rotation of an infinite plane lamina in non-Newtonian liquid: motion started impulsively from rest. Bull. Calcutta Math. Soc. **50** (1958), 57–67.

The authors consider flow of non-Newtonian fluids generated by rotating an infinite plane lamina about an axis parallel to its plane, assuming the fluid is at rest at infinity. Srivastava treats steady flow while Nigam discusses unsteady flow generated by applying a constant rate of rotation impulsively. They note that the effect of cross viscosity seems to be similar to that produced in a Newtonian fluid by permitting the fluid to rotate with constant angular velocity at infinity, about the axis of rotation of the lamina. The theory used is a second order theory of the Reiner-Rivlin type.

J. L. Ericksen (Baltimore, Md.)

1803:

Kontorovich, V. M. On the interaction between small disturbances and discontinuities in magnetohydrodynamics and on the stability of shock waves. Soviet Physics. JETP **35** (8) (1959), 851–858 (1216–1225 Ž. Eksper. Teoret. Fiz.).

Considérant un milieu susceptible d'être décrit par les équations de l'aéromagnéto-dynamique, l'auteur étudie la rencontre d'une onde plane (onde d'Alfven ou onde sonore) avec une surface de discontinuité (onde de choc). Il indique une méthode géometrique simple pour construire les ondes réfractées et réfléchies. La stabilité de l'onde de choc sous l'influence des ondes planes est ensuite étudiée.

H. Cabannes (Marseille)

1804:

Whitham, G. B. Some comments on wave propagation and shock wave structure with application to magneto-hydrodynamics. Comm. Pure Appl. Math. **12** (1959), 113–158.

The author considers an equation like

$$(1) \qquad \prod_{r=1}^{n}\left(\frac{\partial}{\partial t}+c_r\frac{\partial}{\partial x}\right)\phi+\lambda\prod_{s=1}^{m}\left(\frac{\partial}{\partial t}+a_r\frac{\partial}{\partial x}\right)\phi = 0$$

(where $m < n$), which is of order n and has characteristic speeds of propagation $c_1, \cdots, c_n$, but which one might be tempted for large λ to represent by the equation

$$(2) \qquad \prod_{s=1}^{m}\left(\frac{\partial}{\partial t}+a_r\frac{\partial}{\partial x}\right)\phi = 0,$$

of lower order, m, and with characteristic speeds of propagation $a_1, \cdots, a_n$. He shows that, in fact, the system is unstable unless both $m = n-1$ and the inequality

$$c_1 > a_1 > c_2 > a_2 > \cdots > a_{n-1} > c_n$$

holds, but that solutions of (1) do asymptote in this case to solutions of (2) as $\lambda \to \infty$.

The ideas are particularly valuable after extension to non-linear problems; a special case noticed before [Whitham, Proc. Roy. Soc. London Ser. A **219** (1955), 281–316; MR **17**, 912] is concerned with roll-waves (the unstable case) and flood-waves (the stable case), and the author generalises his previous diagnosis of the structure of roll-waves to a wide class of unstable, non-linear problems.

Both the linear and non-linear theories are illustrated by reference to one-dimensional longitudinal wave propagation in a transverse magnetic field.

M. J. Lighthill (Manchester)

1805:

Ludford, G. S. S. The structure of a hydromagnetic shock in steady plane motion. J. Fluid Mech. 5 (1959), 67–80.

The structure of a magnetohydrodynamic shock is discussed within the framework of the usual continuum theory. The normal shock, in which the tangential velocity and normal magnetic field are zero, was studied in detail by Marshall [Proc. Roy. Soc. London Ser. A **233** (1955), 367–376; MR **17**, 921]. This paper considers some aspects of the more general case of an oblique shock.

Particular attention is paid to the case when the tangential component of the magnetic field vanishes on both sides of the shock, since it is found that the transition (for the same values on the two sides) is not unique. In addition to the pure gas dynamic solution, there is a whole family of transitions in which the tangential magnetic field increases from zero and returns to zero in the shock layer. It is claimed that all solutions except the pure gas dynamic one and sometimes one other can be eliminated by appeal to the fact that any realistic solution must be the limit of an axisymmetric one, the important question being how the current lines are closed at infinity. In this particular case, however, the total current going to infinity is zero, so that it is hard to believe that the return current lines can have such an important effect; indeed, in other cases, which are accepted because the structure is unique, the total current is not zero. In detail, the limiting process applied to the axisymmetric case is not convincing and a more natural one does not appear to exclude any of the above solutions.

In the reviewer's opinion, the lack of uniqueness may well be genuine arising, roughly speaking, because in this exceptional case an Alfvèn wave can travel along with the ordinary shock; in the Alfvèn wave the magnetic field increases from zero and returns to zero, so that it is not determined by conditions at infinity.

G. B. Whitham (Cambridge, Mass.)

1806:

Bazer, J.; and Ericson, W. B. Hydromagnetic shocks. Astrophys. J. **129** (1959), 758–785.

Les auteurs résolvent et discutent, dans le cas le plus général, les équations des chocs magnétoaérodynamiques pour un gaz polytropique. La vitesse du choc et les grandeurs qui caractérisent l'état du fluide après le choc sont exprimées en fonction de la discontinuité du champ magnétique; ces fonctions sont algébriques. Les chocs sont classés en fonction de cette représentation.

H. Cabannes (Marseille)

1807:

Germain, Paul. Sur la structure de certaines ondes de choc dans un fluide conducteur en présence d'un champ magnétique. C. R. Acad. Sci. Paris **248** (1959), 1929–1931.

Plane stationary shocks are studied in a fluid with electrical conductivity σ, viscosity m, and thermal conductivity λ, in the presence of a magnetic field perpendicular to the direction of flow. If $\sigma^{-1}=m=\lambda=0$, then a particular flow regime characterised by certain values of pressure, density, velocity, and magnetic field can be connected by a shock to at most one other such flow; the flow velocity is supersonic upstream and subsonic down-stream. For non-vanishing σ^{-1}, m, and λ, the transition through a shock connecting two flow regimes is unique.

A. Herzenberg (Manchester)

1808:

Zhigulev, V. N. On a class of motions in magnetohydromechanics. J. Appl. Math. Mech. **22** (1958), 537–539 (389–390 Prikl. Mat. Meh.).

Motions are considered in a medium with vanishing viscosity and thermal conductivity, and infinite electrical conductivity, under the restriction that neither the magnetic field vector **H** nor the velocity **v** should vary along lines of force. The following results are obtained. (1) If the flow is isentropic and the quantity $|\mathbf{H}|/\rho$ is constant (ρ is the density), then the circulation $\oint \mathbf{v}\cdot d\mathbf{e}'$ taken around a contour moving with the fluid is constant in time. (2) An analogue to Bernoulli's equation is proved if the motion is steady. (3) If the motion is unsteady but irrotational, then $\partial\varphi/\partial t+v^2/2+\int(dp/\rho)+H^2/4\pi\rho=\omega_1(t)$, where φ is the velocity potential, and ω_1 a function of time.

A. Herzenberg (Manchester)

1809:

Ludford, G. S. S. Rayleigh's problem in hydromagnetics: The impulsive motion of a pole-piece. Arch. Rational Mech. Anal. **3** (1959), 14–27.

Rayleigh's problem, namely the flow engendered by the uniform motion, after an impulsive start, of an infinite flat plate immersed in an infinite fluid is generated to include the effects of (i) electrical conductivity, (ii) transverse magnetic field and (iii) a second parallel flat plate.

A formal solution is obtained for general values of the kinematic viscosity ν and the conductivity σ and its properties discussed. In the case $\mu\sigma\nu=1$ (μ is the permeability) the solution is shown to have a particularly simple form.

If $\nu\ll1$, $\sigma\gg1$, the vortex sheets at the plates are shown to be dispersed instantaneously as Alfvén waves travelling with constant velocity and reflected from each plate in turn. The flow pattern is thus reminiscent of that found by the reviewer in a somewhat related problem [Proc. Cambridge Philos. Soc. **53** (1957), 774–775; MR **19**, 354].

K. Stewartson (Durham)

1810:

Pai, S. I. Laminar jet mixing of electrically conducting fluid in a transverse magnetic field. J. Aero/Space Sci. **26** (1959), 254–255.

This note discusses a jet of electrically conducting, incompressible, viscous fluid issuing from a narrow two-dimensional slit. A weak magnetic field is applied perpendicularly to the slit. The stream function is expanded as a power series in the magnetic field, and the induced magnetic field is neglected altogether. A particular rate of decrease of the applied field with distance from the slit is shown to lead to a similar solution whose velocity distribution shape is independent of distance from the slit, the spread of the jet mixing region is derived for this case.

A. Herzenberg (Manchester)

1811:

Fourès-Bruhat, Yvonne. Fluides chargés de conductivité infinie. C. R. Acad. Sci. Paris **248** (1959), 2558–2560.

L'auteur établit les équations des chocs pour un fluide relativiste doué de conductivité électrique infinie. Elle en déduit par considération des chocs infiniment faibles, l'équation qui détermine les hypersurfaces constituant les

fronts d'ondes du premier ordre, et calcule la vitesse de propagation de ces ondes par rapport au repère propre.
H. Cabannes (Marseille)

1812:

Yih, Chia-Shun. Effects of gravitational or electromagnetic fields on fluid motion. Quart. Appl. Math. **16** (1958), 409–415.

For the case of an inviscid, incompressible fluid having negligible magnetic viscosity, the author shows that the effects of gravity or an electromagnetic field are to stiffen the fluid along isopycnic surfaces or lines, or along lines of force, respectively. *C. H. Papas* (Pasadena, Calif.)

1813:

Akhiezer, A. I.; and Sitenko, A. G. Theory of excitation of hydromagnetic waves. Soviet Physics. JETP **35** (8) (1959), 82–85 (116–120 Ž. Eksper. Teoret. Fiz.).

The authors present a theory for the excitation of magneto-hydrodynamic and magneto-acoustic waves by means of external currents. They compare the intensity of these waves with the intensity of waves excited by mechanical means [S. Lundquist, Phys. Rev. (2) **76** (1949), 1805–1809]. *C. H. Papas* (Pasadena, Calif.)

1814:

Korobeinikov, V. P. Similarity-type one-dimensional motions of a conducting gas in a magnetic field. Soviet Physics. Dokl. **121** (3) (1958), 739–742 (613–615 Dokl. Akad. Nauk SSSR).

This paper discusses some general properties of the motions in one dimension or with cylindrical symmetry of a gas with vanishing viscosity and thermal conductivity, and with infinite electrical conductivity in the presence of a magnetic field. If the initial and boundary conditions contain two parameters of independent dimensions ML^kT^s and LT^{-s}, the equations can be reduced to non-dimensional form and simplified, having sets of similar solutions differentiated only by the values of the two parameters. Applications are suggested to motions with given initial conditions, a shock initiated by a piston, and a strong detonation. *A. Herzenberg* (Manchester)

1815:

Kato, Yusuke. Interactions of hydromagnetic waves. Prog. Theoret. Phys. **21** (1959), 409–420.

L'auteur considère des mouvements rectilignes d'un fluide compressible doué de conductivité électrique infinie. Après avoir rapellé les équations de choc, il étudie l'interaction d'une onde de choc et une discontinuité de contact, et l'interaction d'une onde de raréfaction et une discontinuité de contact. *H. Cabannes* (Marseille)

1816:

Naze, Jacqueline. Sur certains écoulements quasi rectilignes d'un fluide doué de conductivité électrique finie. C. R. Acad. Sci. Paris **248** (1959), 525–528.

This note begins with a treatment of the same problem as was studied by Resler and Sears [references given in review below]. Here the "one-dimensional flow" approximations are made in such a way that the current is proportional to $\partial H/\partial x$, where x is the coordinate along the channel and H is the magnetic field strength. Relations are obtained for the velocity as function of x when the cross-sectional area is prescribed. The stability is then studied by the same technique as in the paper reviewed below. Again it is concluded that only decelerating transonic flow is unstable. This stability investigation is subject to the same criticism as in the review below.

The reviewer also disagrees with the treatment of the steady-flow problem and suggests that the meaning of "one-dimensional" has been misconstrued. For the channel of nearly uniform area the current cannot be given by $\partial H_y/\partial x$; in fact the other term of curl **H**, viz. $\partial H_x/\partial y$ is much greater for steady flow. That this latter derivative does not appear in the analysis is due to the one-dimensional approximation, i.e., to the fact that an average has been taken across the section of the channel. Thus the reviewer cannot accept the results of this note nor those of a previous investigation [C. R. Acad. Sci. Paris **246** (1958), 3316–3319] where a fluid of infinite conductivity was considered. For the same reason, exception is taken to the author's description of the Resler-Sears work, viz., "... les fluctuations des champs dues au mouvement du fluids sont négligées; ..." A similar statement also appears in her review [MR **19**, 1226] of the Resler-Sears paper.
W. R. Sears (Ithaca, N.Y.)

1817:

Naze, Jacqueline. Étude de la stabilité des écoulements de Resler-Sears. C. R. Acad. Sci. Paris **248** (1959), 362–365.

In two recent papers [J. Aero. Sci. **25** (1958), 235–246; Z. Angew. Math. Phys. **9b** (1958), 509–518; MR **19**, 1226; **20** #616] Resler and the reviewer studied steady flow of electrically conducting perfect gases in channels of nearly uniform cross section, under the influence of applied electric E and magnetic H fields. This paper represents an attempt to study the stability of such flows. This involves the consideration of waves propagating along the channel. Unfortunately, the author has taken H to be unperturbed by such waves, which is not appropriate for the unsteady case, and therefore arrives at a characteristic velocity (sound speed) independent of H. The correct value has been given by van de Hulst [Proc. Astrophysical Symposium, Paris, 1949] and others. Using this result she has studied the stability in the manner of R. E. Meyer [Quart. J. Mech. Appl. Math. **5** (1952), 257–269; MR **14**, 329]. She concludes that flow decelerating through the sonic speed is unstable while in all other cases studied in the original reference (excepting of course those exhibiting "choking") the electromagnetic effect increases the stability. *W. R. Sears* (Ithaca, N.Y.)

1818:

Gheorghiţă, Şt. I. Sur les mouvements non linéaires dans les milieux poreux. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 491–502. (Romanian. Russian and French summaries)

The author studies fluid motion in porous media typified by non-linear partial differential equations. A procedure, similar to the Janzen-Rayleigh type, is proposed as a possible means of obtaining solutions to these equations. The strong non-linearity is relaxed in certain cases by

means of symmetry considerations. Some examples are presented, and a certain generalization of the equation of Boussinesq is also obtained. It should be noted, however, that the author does not present any rigorous justification of his boundary-conditions.

K. Bhagwandin (Oslo)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 1388, 1723, 1814, 1882.

1819:

Biot, A. Un oculaire triplet pour microscope. Ann. Soc. Sci. Bruxelles. Sér. I **69** (1955), 24–26.

The design of the commonly used Huygens' ocular is modified to increase the distance between the exit-pupil and the eye piece lens. The new type of ocular has various practical advantages. In particular it is more convenient for observers who wear glasses.

E. W. Marchand (Rochester, N.Y.)

1820:

Biot, A. Un oculaire triplet. II. Théorie de Gauss. Ann. Soc. Sci. Bruxelles. Sér. I **72** (1958), 39–48.

In an earlier paper [reviewed above] the author described a triplet constituting of a modified Huygens ocular. The present paper investigates the Gaussian optics of this new lens basing the discussion on classical thin lens laws. Achromatism conditions are derived on the assumption of vision at infinity.

E. W. Marchand (Rochester, N.Y.)

1821:

Biot, A. Un oculaire triplet. III. Condition générale d'achromatisme. Ann. Soc. Sci. Bruxelles. Sér. I **72** (1958), 49–56.

Reference is made to earlier papers by the same author [reviewed above] describing a modified Huygens ocular. The achromatism investigation is generalized to include cases where the image is at any finite distance.

E. W. Marchand (Rochester, N.Y.)

1822:

Koniukov, M. V. On low-frequency oscillations in the plasmas of electronegative gases. Soviet Physics. JETP **5** (1957), 429–432.

The author examines the problem of determining the oscillations of a plasma consisting of a mixture of three ideal gases, viz., an electron gas, a positive-ion gas, and a negative-ion gas. The novelty of the analysis stems from the inclusion of the negative ions, all previous treatments being limited at most to a mixture of electrons and positive ions. Although interactions with a neutral gas and effects of the creation and annihilation of charged particles are assumed to be negligible, the results of the analysis are not compromised because interest is centered on the low-frequency oscillations at relatively low pressures. Two modes of low-frequency oscillations are found which depend differently on the densities of the charged particles.

C. H. Papas (Pasadena, Calif.)

1823:

Rosenbluth, M. N.; and Longmire, C. L. Stability of plasmas confined by magnetic fields. Ann. Physics **1** (1957), 120–140.

The paper examines the question of the stability of plasmas by considering the orbits of individual particles. This approach is complementary to the conventional treatment using magnetohydrodynamic equations. The present results on the whole confirm previous conclusions and in some cases extend their generality. First order orbit theory is discussed in detail, and is then applied to the case when small perturbations occur. The instability is also discussed from a thermodynamic point of view, and the change in the integral energy of the plasma in the case of a flute-type instability is given. Finally two non-linear effects on the flute growth are investigated: the effect of the component of electric field parallel to the magnetic field, and the case when the amplitude exceeds the wave length—a situation very similar to the Taylor instability.

M. J. Moravcsik (Livermore, Calif.)

1824:

Kovrizhnykh, L. M. On the dynamics of a bounded plasma in an external field. Soviet Physics. JETP **6** (1958), 54–58.

The dynamics of a quasi-neutral plasma formation situated in the field of a plane electromagnetic wave are considered. Since rigorous treatment of the problem would be very complicated, a successive approximating method is used. The results hold only when the deformation of the condensation is negligible. The conclusion is that, at least when $ka \ll 1$ (k being the propagation vector of the electromagnetic wave, and a some characteristic dimension of the system), the plasma bunch tends to spread out.

M. J. Moravcsik (Livermore, Calif.)

1825:

Kahn, F. D. Velocity changes of charged particles in a plasma. Astrophys. J. **129** (1959), 468–474.

Mechanisms for the acceleration and deceleration of fast charged particles traversing a plasma are discussed. The deceleration mechanisms are (i) the excitation of a plasma excitation increasing exponentially with time by a stream of fast protons acting coherently, and (ii) the production of a wake in the plasma by single fast particles acting incoherently. The acceleration mechanism is the action of the fluctuating electric field in the plasma. The deceleration mechanism (i) is shown to require a velocity spread of the protons which is much smaller than the difference between electron and proton mean velocities, so much so that the mechanism is unlikely in astronomical applications. Deceleration mechanism (ii) is shown to be sufficiently weak for quite a low level of plasma excitation to accelerate fast particles.

A. Herzenberg (Manchester)

1826:

Nishiyama, Toshiyuki. Electrostatic interactions in an electron-ion gas at high density. Prog. Theoret. Phys. **21** (1959), 389–408.

This paper is a continuation of work done by the author on the properties of electron-ion gas [Prog. Theoret. Phys. **6** (1951), 366–378; **14** (1955), 38–51; MR **13**, 714; **18**, 444]. The electron-ion interaction is studied here by three methods: (a) the method of normal modes, similar to that of Sawada [Phys. Rev. (2) **106** (1957), 372–383; MR **19**, 98]; (b) the sound approximation;

(c) the collective description. Using (a) it is shown that the interaction between electrons and photons corresponding to the Bloch interaction for metals can be derived by eliminating the interaction between the electronic plasma and the ionic plasma. The method (b) turns out to be equivalent to the Gell-Mann-Brueckner method [ibid. **106** (1957), 364–368; MR **19**, 98]. The dispersion equations are also obtained in this approximation. In method (c) the interaction between holes and holes, and between excited electrons and excited electrons are taken into account, and some implications to superconductivity are discussed.

M. J. Moravcsik (Livermore, Calif.)

1827:

Brinkman, H. C. Theoretical aspects of the behavior of a plasma in electric and magnetic fields. Nederl. Tijdschr. Natuurk. **25** (1959), 133–141. (Dutch)

1828:

Wilhelmsson, Hans. The interaction between an obliquely incident plane electromagnetic wave and an electron beam. III. Chalmers Tekn. Högsk. Handl. no. 206, 17 pp. (1958).

A more direct physical interpretation is given for relations previously derived in the formal solution of the interaction between an obliquely incident plane wave and an electron beam. [H. Wilhelmsson, Chalmers Tekn. Högsk. Handl. no. 155 (1954); no. 198 (1958); MR **20** #1522.] It is found that no space charge waves can be excited in the beam in the absence of a static magnetic field. In this case and for an infinitely strong axial magnetic field, the beam acts like a metallic reflector, provided that the axial phase velocity of the incident wave equals the velocity of the beam, and that the magnetic field vector of the incident wave is polarized perpendicularly to the beam. Explicit expressions are obtained for the a.c. components of the current density and the electron velocity.

J. E. Rosenthal (Passaic, N.J.)

1829:

Destouches, Jean-Louis; et Yêm, Pham Xuân. Interactions entre matière et rayonnement en théorie fonctionnelle. C. R. Acad. Sci. Paris **248** (1959), 2959–2961.

Nous nous proposons d'étudier, en théorie fonctionnelle, le mouvement d'un électron dans un champ électromagnétique à un grand nombre de photons.

Résumé des auteurs

1830:

Millar, R. F. Radiation and reception properties of a wide slot in a parallel-plate transmission line. I, II. Canad. J. Phys. **37** (1959), 144–169.

The radiation problem of the title is solved on the condition that the dominant wave is the only propagating wave in the parallel-plate region. In Part I [II] the electric [magnetic] vector is parallel to the edges of the slot. A pair of integral [integrodifferential] equations is set up for the currents in the plate containing the slot. These equations are solved by the Wiener-Hopf technique. Both the reflection and transmission coefficients and the amplitude of the radiated field from the slot are determined. Asymptotic expansions of the various quantities are derived if the width of the slot is large with respect to the free-space wavelength. The reception properties of the system under plane-wave excitation follow from the radiation properties by application of reciprocity theorems. Numerical results are compared with available experimental data.

C. J. Bouwkamp (Eindhoven)

1831:

Lebedev, N. N.; and Skal'skaia, I. P. Axially-symmetric electrostatic problem for a thin-walled conductor in the form of a half-infinite tube. Soviet Physics. Tech. Phys. **28** (4) (1958), 740–748 (792–800 Ž. Tehn. Fiz.).

The author determines the potential of a semi-infinite circular tube in an electrostatic field having symmetry with respect to the axis of the cylinder. This is accomplished by formulating dual integral equations and following the method of Wiener and Hopf.

A. E. Heins (Ann Arbor, Mich.)

1832:

Gáspár, R.; Koltay-Gyarmati, B.; and Tamássy-Lentei, I. Determination of electrostatic potentials by series. Acta Phys. Acad. Sci. Hungar. **9** (1958/59), 369–380.

A treatment is presented for expanding the solution of Poisson's equation $\Delta\phi = -4\pi\rho$, which satisfies prescribed boundary conditions, in terms of the eigenfunctions of $\Delta\phi = E\phi$ satisfying the same boundary conditions. In the general treatment the charge density is due to a point charge. The authors state that the method applies to all problems where the charge density can be expressed as a sum of terms involving the Dirac δ-function. Examples are worked out in which the charge density is constant over prescribed surfaces. It is stated that the convergence of the series is sufficiently rapid.

J. E. Rosenthal (Passaic, N.J.)

1833:

Sturrock, P. A. Kinematics of growing waves. Phys. Rev. (2) **112** (1958), 1488–1503.

The author states that he has two objectives: (1) to define exactly (i.e., mathematically) and to distinguish between the terms "amplifying waves" and "evanescent waves"; (2) to derive criteria for recognizing these waves. His method is to treat the waves "kinematically", that is, as making up a free wave packet in space, or time, or both. An examination of the spatial behavior of a wave packet in time leads to unambiguous definitions and with the interchange of space and time to a clear distinction between two kinds of wave instability. The criteria for recognizing these wave characteristics stem from the dispersion relation associated with the periodic configuration supporting the wave. By means of this relation the representation of the wave function as a Fourier integral over all frequencies is transformed to a corresponding representation over all wave numbers. These representations are then sufficient to determine the space-like and/or time-like nature of the wave packet and thereby its character as a growing wave in accordance with the definitions.

These ideas are illustrated by the detailed analysis of a simple mechanical system and are further elucidated by considering their implications in more general situations.

R. D. Kodis (Providence, R.I.)

1834:

★Wilhelmsson, Hans. The scattering of electromagnetic waves by an electron beam and a dielectric cylinder. Doktorsavhandlingar vid Chalmers Tekniska Högskola, No. 18. Göteborg, 1958. ii + 8 pp.

1835:

Motz, H.; and Nakamura, M. Radiation of an electron in an infinitely long waveguide. Ann. Physics **7** (1959), 84–131.

The authors calculate the radiation emitted by an electron having a prescribed motion in a perfectly conducting endless cylindrical tube filled with a uniform linear medium. They consider three types of motion: (a) uniform motion of the electron parallel to the axis of a tube of arbitrary cross-section, (b) uniform motion with superposed longitudinal oscillations, (c) undulating motion with constant longitudinal velocity.

C. H. Papas (Pasadena, Calif.)

1836:

Talanov, V. I. Generation of surface waves at the open end of a plane waveguide. Soviet Physics. Tech. Phys. **28** (3) (1958), 1185–1195 (1275–1285 Ž. Tehn. Fiz.).

The author considers the interesting problem of determining the reflection and diffraction of a fundamental E-wave traveling toward the open end of a two-dimensional waveguide made up of a reactive surface covering the entire plane $x=0$ and a parallel electric wall lying in the half-plane $x=d$, $z \leq 0$. The boundary condition along $x=0$, which he takes to be $E_z = ZH_y$ where Z is purely reactive, coupled with the boundary condition at the plane $x=d$ and the radiation condition, leads to a couple of integral equations. Using the Wiener-Hopf method he finds solutions of these equations that yield the resulting fields inside and outside the waveguide, the outside field being a surface wave plus a radiation field and the inside field a reflected wave.

C. H. Papas (Pasadena, Calif.)

1837:

Unger, Hans-Georg. Helix waveguide theory and application. Bell System Tech. J. **37** (1958), 1599–1647.

The author analyzes the electromagnetic waves in a "helix waveguide" consisting of a closely wound insulated wire covered with a dielectric jacket and surrounded by a coaxial metallic shield. The telegraphist's equations are used as a point of departure and the analysis is made tractable by invoking the radial wave impedance of the helix interface. Applications of such a waveguide as a mode filter, a transmission medium, and a device to send a circular-electric wave around a sharp bend are fully discussed.

C. H. Papas (Pasadena, Calif.)

1838:

Barlow, H. E. M. Propagation around bends in waveguides. Proc. Inst. Elec. Engrs. C. **106** (1959), no. 9, 11–15.

There are many applications of waveguides in which bends are unavoidable and the problem of assuring the same undisturbed field pattern throughout arises.

In a former paper [same Proc. B **104** (1957), 403–409] the author showed that in the case there considered a good approximation to undisturbed propagation around a bend is obtained by suitable variation with regard to the centre of curvature of the permittivity and the permeability of the medium within the tube. That idea is now extended to various other cases.

For a hollow metal waveguide of rectangular cross section he starts from Maxwell's equations in cylindrical coordinates. For a dielectric coated single wire wave guide he uses toroidal coordinates. In both cases, assuming that the variation in time is sinusoidal, the required solution is obtained in Bessel functions (or sine functions) involving the above mentioned material constants. From the discussion he concludes that it is urgent to develop dielectric materials adapted to these applications.

H. Bremekamp (Delft)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 1458, 1724, 1726, 1788.

1839:

Carroll, Charles W. Systematic approach to the calculation of thermodynamic transforms. Amer. J. Phys. **27** (1959), 302–306.

The various relations which exist between the thermodynamic properties of pure substances are very important, but also very numerous. From time to time authors are tempted to systematize the procedures for deriving the relationships. In this, of course, the use of Jacobians enables one to contract the derivation appreciably.

The reviewer fails to see in what way the present attempt is superior to the others available, or to the one that any competent physicist is likely to develop for himself when the need arises.

J. Kestin (Providence, R.I.)

1840:

Vodička, Václav. A contribution to the unit representation of temperature functions in the case of immovable sources of heat. Arch. Mech. Stos. **10** (1958), 783–792. (Polish and Russian summaries)

Let $u(x, y, z, t)$ denote temperatures in a homogeneous solid of infinite extent in all directions, with constant thermal coefficients. The initial temperature distribution is some prescribed function of the coordinates x, y, and z. Instantaneous heat sources, and sources that vary with time during prescribed intervals, are present on fixed sets of points within the solid. The sets include discrete points, arcs, surfaces and volumes. With the aid of step functions a single formula is written for the temperature u at all points and for all times t. As a special case of his general formula the author notes a compact formula for temperatures in the solid, initially at temperature zero, when the only sources consist of an instantaneous source distributed uniformly along a straight line segment.

R. V. Churchill (Ann Arbor, Mich.)

1841:

Kessler, Arnošt. Die Erwärmung von Platten und endlichen Stäben mit innerer Wärmeerzeugung, bei eindimensionaler Lösung der Wärmeströmung. Apl. Mat. **3** (1958), 190–222. (Czech. Russian and German summaries)

The author considers one dimensional heat conduction in a rod or a plate of finite length with internal heat sources. The heat generated internally is either constant or linearly dependent on the temperature. The author assumes zero initial temperature and three different kinds of boundary conditions. He presents a detailed derivation of the partial differential equation controlling the phenomenon, which he could have omitted, because the result is the well known one-dimensional heat equation with an additional term representing the internally generated heat. The boundary value problem is solved by the Laplace transformation and the results are presented in dimensionless form. *T. Leser* (Aberdeen, Md.)

QUANTUM MECHANICS

See also 1343, 1545, 1635, 1725, 1826.

1842:

Blokhintsev, D. I. Some remarks on the validity of the hydrodynamic description of quantum systems. Soviet Physics. JETP **5** (1957), 286–287.

A hydrodynamic description of motion assumes that it is possible to assign to each element of the medium an energy density, a momentum density, a medium density and a pressure. From these the author constructs the momentum of an element and the momentum dispersion (containing the Planck constant). To describe the motion, it is required that the mean value of the momentum be larger than the possible dispersion. The inequalities obtained are applied to describe the atomic nucleus and multiple creation of mesons. In the first case the author shows that the moment of inertia of a nucleus computed from the motion of an ideal liquid has no relation to the actual situation. The second case shows that the hydrodynamic description is absolutely inapplicable.

M. Z. v. Krzywoblocki (Urbana, Ill.)

1843:

Phan-Van-Loc. Interprétation physique de l'expression mathématique du principe de Huygens en théorie de l'électron de Dirac. Cahiers de Phys. **12** (1958), 327–340.

On the analogy of the development of Kirchoff's formula, the solution of Dirac's equation for an electron is expressed as the sum of a surface and a line integral which correspond, respectively, to effects due (i) to the motion of the electron as a whole and (ii) to its electric and magnetic moments. *A. J. Coleman* (Toronto, Ont.)

1844:

Ullmo, J. Théorie des probabilités et mécanique quantique. Publ. Inst. Statist. Univ. Paris **6** (1957), 251–258.

A few remarks are made on a recent paper by Heisenberg [in *Niels Bohr and the Development of Physics*, Pergamon Press, London; McGraw-Hill, New York; 1955; MR **17**, 218] on the interpretation of quantum theory. They suggest that Heisenberg's views do not necessarily reject more deterministic interpretations of the theory.

L. Van Hove (Utrecht)

1845a:

Olsson, Per O. M. On Coulomb wave functions. Ark. Fys. **15** (1959), 131–143.

1845b:

Olsson, Per O. M. On transition matrix elements between states in a Coulomb field. I. Ark. Fys. **15** (1959), 159–167.

The calculation of matrix elements for radiative transitions between various states of a Coulomb field is a problem of great technical complexity. This is essentially due to the fact that the relativistic continuum wave functions contain confluent hypergeometric functions and the requisite integrals are difficult to evaluate. In an effort to simplify this problem, a great deal of work has been devoted to heuristic truncations of the exact wave functions, sufficiently drastic to confine the results of all subsequent calculations to a comparatively small area of 'known' or 'readily computable' special functions. Recently the Scandinavian school has sought to redress the balance somewhat through renewed attention to the possibilities of 'exact' methods. The present papers are part of a series [see also K. Alder and A. Winther, Danske Vid. Selsk. Mat.-Fys. Medd. **29** (1955), no. 18, no. 19; MR **17**, 566] devoted to assembling and adapting some results from the theory of generalized hypergeometric functions for possible use in the calculation of these Coulomb problems.

The first paper contains a rederivation of the relativistic continuum wave functions for the Coulomb field where, however, the solutions are written in terms of the confluent Appell function Φ_2 instead of the usual ${}_1F_1$. The principal replacement is

$$e^{-ipr}{}_1F_1(\gamma+i\eta;\,2\gamma;\,2ipr) = \Phi_2(\gamma+i\eta,\,\gamma-i\eta;\,2\gamma;\,ipr,\,-ipr).$$

It is claimed that many useful transformations—in particular, analytic continuation—are more transparently derivable from the new Φ_2 form. [For details of the notation see P. Appell and J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques*, Gauthier-Villars, Paris, 1926.]

The second paper illustrates this point in further detail by considering one of the radial integrations typical of an internal conversion calculation; the old approach is as follows:

$$(1)\quad \int_0^\infty e^{-(\mu-ik+ip)r}r^{q-1}{}_1F_1(\gamma+i\eta;\,2\gamma;\,2ipr)dr = \frac{\Gamma(q)}{(\mu-ik+ip)^q}\,{}_2F_1\left(q,\,\gamma+i\eta;\,2\gamma;\,\frac{2ip}{\mu-ik+ip}\right).$$

With this method the further calculations are seriously hampered because the (often useful) series development of ${}_2F_1$ requires

$$\left|\frac{2p}{\mu-ik\pm ip}\right| < 1$$

which is too stringent a condition for many cases of physical interest. It is possible to palliate this difficulty by resorting to transformations of the last place of ${}_2F_1$ through analytic continuation, but this procedure yields cumbersome expressions. With the aid of his new formulation for the continuum wave functions, Olsson now

replaces (1) by the equivalent

$$\int_0^\infty e^{-(\mu-ik)r}r^{q-1}\Phi_2(\gamma+i\eta,\gamma-i\eta\,;2\gamma\,;ipr,-ipr)dr =$$

$$\frac{\Gamma(q)}{\mu^q}\left(\frac{\mu-i(k+p)}{\mu}\right)^{-\gamma-i\eta}\left(\frac{\mu-i(k-p)}{\mu}\right)^{-\gamma+i\eta}$$

$$\times F_1\left(2\gamma-q,\gamma+i\eta,\gamma-i\eta\,;2\gamma\,;\frac{i(k+p)}{i(k+p)-\mu},\frac{i(k-p)}{i(k-p)-\mu}\right),$$

which, it is asserted, is far more suited to numerical work inasmuch as the double series representation of F_1 converges for all cases of physical interest.

T. Erber (Chicago, Ill.)

1846:

Landé, Alfred. Zur Quantentheorie der Messung. Z. Physik **153** (1958), 389–393 (1959).

Heisenberg and von Neumann's description of the measuring process rests on a double meaning of the term "state" and on a misinterpretation of the Schrödinger equation which further leads to the mixture of subjective and objective elements, the transition from the possible to the factual, etc. If one eliminates the double meaning and consistently accepts the statistical interpretation, one can return from the mixture of physics and phenomenology of subjective elements of consciousness to pure physics again. The remedy of removing intrinsic contradictions from the theory of measurement is just opposite to that of Schrödinger and Feyerabend [cf. P. Z. Feyerabend, same Z. **148** (1957), 551–559; MR **19**, 500]; it is purely statistical [cf. Proc. 9th Symposium Colston Research Soc., Univ. of Bristol, 1957, Butterworths Scientific Publ., London, 1957; A. Landé, Phys. Rev. (2) **108** (1957), 891–893; Amer. J. Phys. **24** (1956), 56–59; MR **20** #6284].

M. Pinl (Cologne)

1847:

Petiau, G. Sur une généralisation non linéaire de la mécanique ondulatoire et les propriétés des fonctions d'ondes correspondantes. Nuovo Cimento (10) **9** (1958), supplemento, 542–568.

Let $W^2-p^2c^2=(m_0c^2)^2\equiv(\mu_0\hbar c)^2$, where m_0 is the rest mass. Solutions of the Klein-Gordon equation $(\Box+\mu_0^2)\psi=0$ are considered for which either $W^2\leqq 0$, $p^2<0$, or $W^2\geqq 0$, $p^2>0$, and which are of the form $\psi_1(t)\psi_2(x,y,z)$. The generalization of the circular and hyperbolic functions in $\psi_1(t)$ to the elliptic functions of Jacobi leads to the consideration of the equation $\Box\psi+\alpha\psi+\gamma\psi^3=0$. The solutions of this equation which correspond to plane waves are studied in some detail. The addition theorems of elliptic functions permit compositions of these solutions and in a certain sense replace the linear superposition principle for solutions of the Klein-Gordon equation.

F. Rohrlich (Iowa City, Iowa)

1848:

Kurdgelaidze, D. F. Periodic solutions of the nonlinear generalized Dirac equation. Soviet Physics. JETP **34** (7) (1958), 1093–1096 (1587–1592 Ž. Eksper. Teoret. Fiz.).

A rather general nonlinear term is introduced into Dirac's equation and the solution is obtained by quadratures. The associated Klein equation is easily obtained and has a nonlinear term. The converse problem of finding a Dirac equation associated with a Klein equation of this type requires the solution of a complicated system of nonlinear ordinary differential equations.

A. J. Coleman (Toronto, Ont.)

1849:

Symanzik, K. Dispersion relations and vertex properties in perturbation theory. Progr. Theoret. Phys. **20** (1958), 690–702.

With the aid of a method developed earlier by Y. Nambu [Nuovo Cimento **6** (1957), 1064–1083; MR **19**, 921] the author investigates the perturbation theory expressions for various scattering amplitudes. In this way he is able to show that a dispersion relation holds, e.g., for nucleon-nucleon scattering amplitudes in perturbation theory, even if this cannot be proved with the aid of the causality condition alone. An interesting question not yet answered is whether the large analyticity domains found in perturbation theory are due to the expansions used, and/or the particular interactions studied, or are consequences of some general principle not yet understood.

G. Källén (Lund)

1850:

Nakanishi, Noboru. On the validity of dispersion relations in perturbation theory. Progr. Theoret. Phys. **21** (1959), 135–150.

Dispersion relations for a number of scattering processes are proved in finite-order perturbation theory, using a theorem of the author's [N. Nakanishi, same Prog. **17** (1957), 401–418; MR **18**, 853] about the properties of perturbation theory integrals when their external momenta form a Euclidean set (somewhat similar techniques have been used by Nambu and Symanzik). Non-forward scattering is included by using a trick of Symanzik [see above review].

For other processes, like Σ-nucleon scattering, there is an abnormal threshold of the type found by Karplus, Sommerfield and Wichmann [Phys. Rev. (2) **111** (1958), 1187–1190; MR **20** #2996]; and therefore dispersion relations cannot be of the normal kind, and must have, at least, a more difficult unphysical region.

J. C. Taylor (London)

1851:

Erber, Thomas. Coherent Compton amplitude at high energies. Ann. Physics **6** (1959), 319–340.

The dispersion relation which expresses the real part of the coherent forward Compton scattering amplitude as an integral over the corresponding total cross section is studied. The aim is a proof that the scattering amplitude is uniformly bounded in the high energy limit. It is accomplished by use of various asymptotic representations and inequalities for the matrix elements between Dirac Coulomb wave functions. The result establishes that each side of the dispersion relation is bounded and that, in particular, the photoelectric cross section and the pair-production (by the nucleus) cross section with the electron in a bound s-state become equal and proportional to E^{-1} for high energies.

F. Rohrlich (Iowa City, Iowa)

1852:

Banerjee, H. On Sommerfeld's approximation in high energy photoelectric effect and one quantum annihilation

of positrons in the *K*-shell. Nuovo Cimento (10) **10** (1958), 863–880. (Italian summary)

The Sommerfeld-Maue (approximate) wave functions for the scattering state of a relativistic electron are employed to compute the angular distribution and total cross section for the photoelectric effect and for one-quantum annihilation of positrons in the *K*-shell. This is a high energy approximation, the error being of relative order $Z/(137\varepsilon)$ where ε is the energy of the scattering state in units of the electron rest energy. No marked qualitative difference from Sauter's formula was found.

F. Rohrlich (Iowa City, Iowa)

1853:

Newton, Roger G. Inelastic scattering. Ann. Physics **4** (1958), 29–56.

The author considers the scattering of a particle by a bound system. By considering the scattering of an electron by a hydrogen atom he is led to the following matrix equation:

$$-\tfrac{1}{2}\hbar^2\nabla^2\Psi M^{-1}+\Psi V=\Psi\mathscr{E}.$$

Here Ψ is a single row matrix, M^{-1} is a diagonal square matrix with constant elements and the potential V is a square matrix. The diagonal matrix $\mathscr{E}$ equals $(E-E_\alpha)\delta_{\alpha\beta}$ where E is the energy and E_α is the energy of one of the excited states of the struck system. This equation is first reduced to its partial waves by the expansion of each element of Ψ in a series of Legendre functions. The resultant equation for the amplitudes can again be formulated as a matrix equation. The S matrix can then be formally evaluated in terms of the properties of the regular and irregular solutions of this equation. The following cases are then investigated: (1) energy too low for inelastic scattering; (2) threshold behavior where there is possibility of cusps in the cross-section as a function of energy; (3) resonance scattering leading to the Breit-Wigner formula.

H. Feshbach (Cambridge, Mass.)

1854:

Moe, Mildred; and Saxon, David S. Variational methods in scattering problems. Phys. Rev. (2) **111** (1958), 950–957.

Various formulations of variational principles for the phase shift and for the scattering amplitude which depend only upon the differential operator of the Schroedinger equation are developed. These are expressed in an amplitude independent form. The generalization depends upon the following remark. Let variation of Q be zero; take the variational form for Q to be $H(Z_1, Z_2, \cdots, Z_n)$ where the Z_i are variational parameters. Then the variational equation is

$$\delta Q=0=\sum\frac{\partial H}{\partial Z_i}\,\delta Z_i$$

with solutions $Z=Z_{i0}$, $Q=Q_0$. If, as in the case of scattering, the δZ_i are related, it becomes possible to set up a variety of functions H which for the exact solution yield the correct value of Q.

H. Feshbach (Cambridge, Mass.)

1855:

Kolsrud, Marius. Variational methods for scattering problems. Phys. Rev. (2) **112** (1958), 1436–1437.

The phase shift for the scattering of a particle by a central field is proportional to the inner product (u, v) where v satisfies the integral equation $v-kv=a$, the kernel of K being symmetric and real. New variational expressions for (u, v) are derived from the iterates of this equation obtained by operating on both sides with powers of K. Similar results are obtained for the scattering amplitude. Numerical examples are given.

H. Feshbach (Cambridge, Mass.)

1856:

Lee, T. D.; Low, F. E.; and Pines, D. The motion of slow electrons in a polar crystal. Phys. Rev. (2) **90** (1953), 297–302.

A variational technique is developed to investigate the low-lying energy levels of a conduction electron in a polar crystal. Because of the strong interaction between the electron and the longitudinal optical mode of the lattice vibrations, perturbation-theoretic methods are inapplicable. Our variational technique, which is closely related to the "intermediate coupling" method introduced by Tomonaga, is equivalent to a simple canonical transformation. The use of this transformation enables us to obtain the wave functions and energy levels quite simply. Because the recoil of the electron introduces a correlation between the emission of successive virtual phonons by the electron, our approximation, in which this correlation is neglected, breaks down for very strong electron-phonon coupling. The validity of our approximation is investigated and corrections are found to be small for coupling strengths occurring in typical polar crystals.

Authors' summary

1857:

Lee, Tsung-Dao; and Pines, David. Interaction of a nonrelativistic particle with a scalar field with application to slow electrons in polar crystals. Phys. Rev. (2) **92** (1953), 883–889.

A general variational technique is developed to study the effect of recoil on the motion of a nonrelativistic particle in a scalar field. The ground-state energy is determined, and the results obtained are shown to be exact in the limit of both weak and strong field-particle coupling. This method is applied to investigate the low-lying energy levels of a conduction electron in a polar crystal. The ground-state energy and effective mass so obtained are shown to be in good agreement with the results of Lee, Low, and Pines [abstracted above] for the intermediate coupling strengths occurring in real polar crystals.

Authors' summary

1858:

Chew, G. F.; and Low, F. E. Effective-range approach to the low-energy *p*-wave pion-nucleon interaction. Phys. Rev. (2) **101** (1956), 1570–1579.

The theory of *p*-wave pion-nucleon scattering is reexamined using the formalism recently proposed by one of the authors (F.E.L.). On the basis of the cut-off Yukawa theory without nuclear recoil it is found, for not too high values of the coupling constant, that: (a) For each *p*-wave phase shift a certain function of the cotangent should be approximately linear at low energies and should extrapolate to the Born approximation at zero total energy. The value of the renormalized unrationalized coupling constant determined in this way from experiment is $f^2=0.08$. A special feature of the predicted energy de-

pendence of the phase shifts is that δ_{33} is positive and the other p phase shifts are negative. (b) The so-called "crossing theorem" requires a relation between the four p phase shifts, so that in addition to the coupling constant only two further constants are needed to completely specify the low-energy behavior. (c) The direction of the energy variation in the (3,3) state is such that a resonance will occur for a sufficiently large cut-off $\omega_{\max}$. Rough estimates indicate that $\omega_{\max} \approx 6$ will produce a resonance at the energy required by experiment. It is argued that the results (a) and (b) are very probably also consequences of a relativistic theory but that (c) may not be.

Authors' summary

1859:

Osada, Jun'ichi; and Fujino, Haruyuki. Effect of the non-relativistic recoil of a source particle in quantum field theories. Progr. Theoret. Phys. **20** (1958), 487–504.

The authors develop a method for calculating effects of the non-relativistic recoil of a source particle in strong interaction with a quantized field. Its starting point is the Chew-Low [see review above] treatment of the static approximation. They show how to obtain corrections to scattering amplitudes and to the effective mass of the particle. The method leads to a scattering amplitude for a neutral scalar meson in agreement with the result of McVoy and Steinwedel [Nuclear Phys. **1** (1956), 164–179; MR **19**, 221; p. 165] and to an effective polaron mass which agrees with that of Lee, Low and Pines [#1856 above]. As further applications they discuss the P-wave scattering of a charged scalar and of a pair-theory meson.

P. W. Higgs (London)

1860:

Edwards, S. F.; and Matthews, P. T. A simple treatment of meson-nucleon scattering. Phil. Mag. (8) **2** (1957), 176–181.

This is a very simple and elegant treatment of the problem of the scattering of pions by nucleons. The pion-nucleon "potential" is calculated in Born approximation and the scattering in this potential is solved exactly. The treatment is nonrelativistic, but the relativistic aspects of "crossing symmetry" and the s-wave interaction are retained. The essential features of low energy pion-nucleon phenomena can be understood, reproducing the success of previous attempts. [Sartori and Wataghin, Nuovo Cimento **12** (1954), 145–147; Chew and Low, #1858 above]. The isotopic spin splitting of the s-wave phase shifts cannot be reproduced, even by considering the contribution from strongly coupled heavy mesons.

E. C. G. Sudarshan (Cambridge, Mass.)

1861:

Salam, Abdus. Recent developments in field theory. Rev. Un. Mat. Argentina **18** (1958), 96–105.

An expository review, in a rather staccato style, by one of the chief workers in quantum field theory, which summarizes "the faith of the modern field theorist" in four axioms pertaining to (1) the non-existence of negative energy states of the total field, (2) the definition and transformation properties of field operators, (3) the commutativity of operators at events separated by a space-like interval, (4) the description of the scattering matrix by the asymptotic condition, replacing the use of the interaction representation. The fundamental nature of Kallén's work, giving expressions for the renormalization constants and proving that Z_3 is essentially infinite, is emphasized.

A. J. Coleman (Toronto, Ont.)

1862:

Davison, B. On the rate of convergence of the spherical-harmonics method in the Milne problem. Canad. J. Phys. **36** (1958), 1323–1335.

"The extrapolated end point, z_0, and the ratio of current to flux at the free surface, $j(0)/f(0)$, for the Milne problem are evaluated by the spherical-harmonics method in the P_N approximation, N being odd. It is shown that, for N large, the approximations to these quantities are related to the exact values by

$$\text{(A)}\quad z_{0,N} = z_0 - \pi^2 c/48N^2 + O\{(c^2+1)/N^3\} + O[N(\zeta_1/\zeta_2)^N f_1(c)],$$

$$\text{(B)}\quad [\psi_1(0)/\psi_0(0)]_N = [j(0)/\rho(0)]\{1+O(\zeta_1/\zeta_2)^N f_2(c)\}\},$$

in which c is the number of secondaries per collision, $f_1(c)$ and $f_2(c)$ are independent of N, and ζ_1 and ζ_2 are the absolutely lesser and greater roots of

$$\zeta^2 - 2\zeta + \kappa^2 = 0,$$

κ being the exact inverse diffusion length. The relation (A) is subject to the restrictions

$$N \gg c \quad \text{if} \quad c > 1,$$

$$N(|\zeta_1/\zeta_2|)^N \ll 1 \quad \text{if} \quad c < 1.\text{"}$$ (Author's abstract)

D. J. Hofsommer (Amsterdam)

1863:

Judd, B. R. The matrix elements of tensor operators for configurations of three equivalent electrons. Proc. Roy. Soc. London Ser. A **250** (1959), 562–574.

The fractional parentage coefficients are computed which relate the configurations l^3 and l^2, with special emphasis on the case $l=3$ (f-electrons). These coefficients were defined by Racah [Phys. Rev. (2) **63** (1943), 367–382]. The configuration f^3 is uniquely characterized by the set of quantum numbers W and U in addition to the usual $SLJJ_z$. These sets refer to the irreducible representations of the rotation group in seven dimensions R_7, and to its subgroup G_2, respectively. With the aid of the fractional parentage coefficients the matrix elements between all states of f^3 of the irreducible tensor operator $U^k = \sum_{i=1}^{3} u^k(i)$, where u^k is defined by its reduced matrix element $(l\|u^k\|l)=1$, are tabulated for $k=2$, 4, and 6. The results seem to indicate that "a hidden mathematical substructure remains to be discovered, one where group theoretical considerations will receive greater emphasis." Applications to the spectrum of $NdCl_3$ are in preparation.

F. Rohrlich (Iowa City, Iowa)

1864:

Foland, W. D.; and Present, R. D. Hydrodynamic theory of spontaneous fission. Phys. Rev. (2) **113** (1959), 613–621.

The penetration factor for spontaneous fission is calculated from the formula

$$P = \exp\left\{-(2/\hbar)\int_{a_2'}^{a_2''} [2m^*(V(a_2)-E)]^{1/2} da_2\right\},$$

where the radius of the nucleus is given by $r = R_0[1+a_2P_2(\cos\theta)]$. The potential energy $V(a_2)$ is calculated

from the Coulomb and hydrodynamic surface energies, m^* is an effective mass such that the kinetic energy is $T=\frac{1}{2}m^*(da_2/dt)^2$, calculated on the assumption of irrotational incompressible hydrodynamic flow, a_2' and a_2'' are the limits of the barrier. The absolute values of the lifetimes are unreliable, but the quantity $d \log_{10} T/d\varepsilon$ (where ε is the barrier height) is found to agree with a value found by Swiatecki [same Rev. **100** (1955), 937–938] from an analysis of experiments at $Z=100$ (T is the lifetime).

A. Herzenberg (Manchester)

1865:

Fronsdal, Christian. Unitary irreducible representations of the Lorentz group. Phys. Rev. (2) **113** (1959), 1367–1374.

"The unitary irreducible representations of the inhomogeneous, proper Lorentz group are determined, using a prescription given by Wigner, with special emphasis on the case of zero rest mass. The principal results are: (a) the construction of one-component representations for the case of zero mass and discrete spin; (b) the existence of a Foldy-Wouthuysen transformation for zero mass and spin $\frac{1}{2}$; (c) the construction of "position operators" for zero mass and spins $\frac{1}{2}$, 1; (d) the complete synthesis of the Dirac, Majorana and Maxwell one-particle theories." (Author's summary)

A. C. Hurley (Melbourne)

1866:

★Ericson, Torleif. Some statistical properties of excited nuclei. Inaugural dissertation. University of Lund, Lund, 1958. 5 pp.

1867:

Datta Majumdar, S. Energy levels of polyatomic molecules. Proc. Phys. Soc. **72** (1958), 635–648.

"The mathematical theory of semi-rigid asymmetric top molecules is re-formulated by introducing a new set of operators for angular momentum. All matrix equations are thereby replaced by ordinary differential equations, and the necessity of solving a formidable secular equation for the determination of energy is dispensed with. An analytic expression for energy correct to the second order can be obtained by the methods delineated here." (Author's abstract)

A. C. Hurley (Melbourne)

1868:

Rosen, S. P. A spherical wave expansion for double β-decay. Canad. J. Phys. **37** (1959), 780–785.

1869:

Ivanenko, D.; and Brodskiĭ, A. On non-linear theory of elementary particles. Dokl. Akad. Nauk SSSR **120** (1958), 995–998. (Russian)

Die Verfasser betrachten nichtlineare Verallgemeinerungen der Dirac-Gleichung für ein gequanteltes Spinorfeld. Sie verwenden dabei eine in einer früheren Arbeit entwickelte Methode zur Bildung von Invarianten [Methode der "Kontraktion", D. Ivanenko und A. Brodskiĭ, Ž. Eksper. Teoret. Fiz. **24** (1953), 383–388; Dokl. Akad. Nauk SSSR **84** (1952), 683–686] welcher die Vorstellung einer universellen Wechselwirkung der Fermionen zu Grunde liegt und die Beschreibung der Materie mit Hilfe eines einzigen "Weltspinors". Für die geplante Verallgemeinerung der Theorie werden zunächst die Bedingungen der gewöhnlichen Theorie gestellt. In die nichtlinearen Terme werden keine Ableitungen eingeführt. Dann kann die Lagrange-Funktion $\mathscr{L}$ des Problems in der bereits in der früheren Arbeit gewonnenen Gestalt übernommen werden und die Verfasser verwenden auch die gewöhnlichen (dreidimensionalen) Vertauschungsbeziehungen für freie Felder. Damit gelingt eine Reduktion der nichtlinearen Summanden und eine Verminderung der Anzahl der unabhängigen Konstanten, wodurch der nichtlineare Teil der Lagrange-Funktion gegenüber den (mit Lorentztransformationen vertauschbaren) Pauli-Transformationen vertauschbar wird. Diese Umformungen sind isomorph zur Spinordarstellung der Drehungsgruppe. Zur Auswahl der Invarianten verwenden die Verfasser einen 8-Komponentenspinor ψ, dessen Darstellungen bei Lorentz-Transformationen und Pauli-Transformationen angegeben werden. Der Spinor ψ läßt sich in eine von R. Feynman [Proc. 7-th Rochester Conference, 1957, Interscience, New York, 1958; MR **20** #4433; part IX, pp. 42–44] angegebene Gestalt setzen, welche die Darstellungen im Isotopenraum in Evidenz setzt. Dabei wird der Isotopenzweikomponentenspinor aus dem Proton-Antineutron (bzw. Antiproton-Neutron) gebildet. — Mit den so gewonnenen formalen Hilfsmitteln entwickeln die Verfasser weiterhin eine Theorie Greenscher Einflußfunktionen, welche im nichtlinearen Falle beim Fehlen äußerer Felder zur linearen Theorie bei vorhandenen Wechselwirkungen (mit dem Boson-Vakuum) aequivalent ist.

M. Pinl (Cologne)

1870:

Andersén, Evert; and Uhlhorn, Ulf. Approach to the quantum mechanical many-body problem with strong two-particle interaction. I. Ark. Fys. **13** (1958), 165–176.

Instead of Brueckner's approach to the nuclear many-body problem—which is essentially a perturbation theory approach—a variational approach which is an extension of the Hartree-Fock method is used. Various trial wave functions are discussed and special attention is paid to the one-dimensional problem of two interacting particles in a common potential field for which an oscillator potential is used.

D. ter Haar (Oxford)

1871:

Andersén, Evert. Approach to the quantum mechanical many-body problem with strong two-particle interaction. II. Ark. Fys. **15** (1959), 181–192.

A one-dimensional system of fermions with Gaussian interactions is discussed along the lines of the preceding paper of this series, and also a system consisting of four three-dimensional fermions. The connection with the nuclear many-body problem is also discussed.

D. ter Haar (Oxford)

1872:

Demkov, Yu. N. The virial law in the theory of collisions. Vestnik Leningrad. Univ. **13** (1958), no. 22, 34–41. (Russian. English summary)

"It is shown that for any problem in the quantum

theory of collisions using Fock's method of scaling it is possible to deduce the virial theorem for the derivation of phase or amplitude of scattering with respect to energy. A generalized variational principle for the collisions of compound systems is given. The correspondence of the formulations of the virial theorem for the bound states and for the continuous spectrum is considered." (Author's summary) *R. M. Evan-Iwanowski* (Syracuse, N.Y.)

1873:

Clark, John W.; and Feenberg, Eugene. Simplified treatment for strong short-range repulsions in N-particle systems. I. General theory. Phys. Rev. (2) **113** (1959), 388–399.

"A new variational approach is developed for studying the properties of systems of particles interacting through singular short-range repulsions that give rise to strong two-particle correlations. The correlated trial function $\Psi_\gamma = e^S\Phi_\gamma$ (state γ) results, with proper choice of S, in a simple form for the energy expectation value $\langle H\rangle$—as well as for other matrix elements of interest—which is devoid of all reference to the strong repulsions except through e^{2S} factors and hence is particularly suited to calculation. In many cases an independent-particle type Φ_γ seems appropriate. The cluster evaluation of this form for $\langle H\rangle$ is discussed, both in the few-particle and many-particle cases. Using the techniques of Iwamoto and Yamada, simplified convergent cluster expansions for the energy expectation value are derived for many-fermion and many-boson systems. A program for application of this method to nuclear problems is being initiated." (Authors' summary)

F. Rohrlich (Iowa City, Iowa)

1874:

Zubarev, D. N.; and Tserkovnikov, Iu. A. Thermodynamics of superconductors. Soviet Physics. Dokl. **122** (3) (1958), 986–990 (999–1002 Dokl. Akad. Nauk SSSR).

The thermodynamics of superconductors is calculated from the Fröhlich Hamiltonian using a statistical perturbation theory and retaining terms to the second order in the electric-phonon coupling constant. Bogoliubov's transformation is used and the undetermined parameters are determined, as usual, by the condition that "dangerous" denominators should not appear in a certain order.

L. N. Cooper (Providence, R.I.)

1875:

Belyaev, S. T. Effect of pairing correlations on nuclear properties. Mat.-Fys. Medd. Danske Vid. Selsk. **31** no. 11, 55 pp.

A recent method introduced in the theory of superconductivity—that of generalizing the fermi gas by introducing correlations between time reversed states—is applied to a finite nucleus. It is assumed that the "superconducting" or "superfluid" pair correlation exists (the question of whether with actual nuclear potentials such a correlation will occur is not treated), and the consequences of this correlation for nuclear properties is examined for an idealized nucleus. The success of this model in explaining a considerable number of experimental facts suggests that such a correlation may exist in some nuclei.

L. N. Cooper (Providence, R.I.)

1876:

Marziani, Marziano. Un teorema di unicità dello stato superconduttivo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **25** (1958), 184–188.

RELATIVITY

See also 1811, 1847.

1877:

Builder, G. The constancy of the velocity of light. Austral. J. Phys. **2** (1958), 457–480.

The author discusses the principle of the constancy of the velocity of light from a historical-critical point of view. In Special Relativity the principle must be interpreted as referring to the average speed measured in an inertial reference system over a forward-and-return path, and it asserts that this speed has the same value c irrespective of the direction of transmission of the light and of the motion of the source. The author criticizes Whittaker, Sommerfeld and others for arguing that a ballistic theory of light emission could be reconciled with Special Relativity, on the grounds that the statement that a corpuscle moving with speed c relative to its source would have the same speed c relative to any observer has no definite meaning unless the reference system is specified. He concludes by stressing the conventional nature of the measurement procedures in Special Relativity and maintains that absolute space and time remain essential to our physical description of the universe. *G. J. Whitrow* (London)

1878:

Jeffreys, H. The clock paradox in special relativity. Austral. J. Phys. **2** (1958), 583–586.

The author criticizes both the analysis of the clock paradox given by G. Builder [same J. **10** (1957), 246–252; MR **19**, 813] and H. Dingle's criticism of it [ibid. **10** (1957), 418–423; MR **19**, 814]. The postulated invariance of $ds=0$ and of the equations $d^2x/dt^2=d^2y/dt^2=d^2z/dt^2=0$—which pick out a class of unaccelerated observers—lead to the condition $ds'=kds$, where k is a constant for a given pair of unaccelerated observers. (The further postulate that $k=1$ requires some comparison of scales.) In the situation usually contemplated in discussions of the clock paradox, Dingle's result would follow only if $k=(1-u^2/c^2)^{-1/2}$, where u denotes relative velocity, and Builder's only if $k=1$. Dingle's result might be right if the accelerated clock were made of the most rigid material hypothetically possible (subject to the condition that the velocities of elastic waves do not exceed c); but will not follow for actual materials. Builder's result cannot follow in any case.

G. J. Whitrow (London)

1879:

Fahy, E. F. The clock paradox in relativity. Austral. J. Phys. **2** (1958), 586–587.

The author suggests a possible observational test in connection with rival points of view concerning the clock paradox. A terrestrial observer of a star lying approximately in the plane of the Earth's orbit would expect to find that any visible line in the star's spectrum oscillates with an amplitude of about 0.5 angstrom and a period of a

year. The author argues that the point of view and analysis put forward by H. Dingle [Nature **180** (1957), 1275–1276; letter to the editor] would imply that these oscillations would be very much out of phase with the Earth's motion, the phase difference being, in general, not an integral number of revolutions. An effect of this kind should be observable, since it means that the oscillations in different stellar spectra would, in general, be out of phase with one another. The other point of view does not imply any such phase differences.

G. J. Whitrow (London)

1880:

Payne, W. T. Spinor theory and relativity. II. Amer. J. Phys. **27** (1959), 318–328.

This is the continuation of a previous paper by the author [same J. **23** (1955) 526–536; MR **17**, 306]. It is an elementary exposition of the theory of second rank spinors, in which the author seeks to avoid as far as possible a notation in terms of components and dotted indices, though a variety of dots and asterisks appear instead. Geometrical representations in some special cases are provided. In a final section the theory is applied to Dirac's and Maxwell's equations.

H. A. Buchdahl (Princeton, N.J.)

1881:

Schiff, L. I. Gravitational properties of antimatter. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 69–80.

The motivation of this paper is the idea that a difference in sign between the gravitational properties of matter and antimatter could lead to their separation on a cosmological scale [P. Morrison and T. Gold, *Essays on gravity*, Gravity Research Foundation, New Boston, N.H., 1957; p. 45; P. Morrison, Amer. J. Phys. **26** (1958), 358–368; D. Matz and F. A. Kaempffer, Bull. Amer. Phys. Soc. **3** (1958), 317. The author shows, on the basis of Eötvös' experiments, that the virtual positrons produced by atoms have in fact the same passive gravitational mass as electrons to within about 1 per cent (unless some fortuitous cancellation occurs). The contribution of virtual antinucleons to the gravitational mass of an atom should be much greater than that of virtual positrons, but the author does not calculate this contribution, since the structure of meson-nucleon field theory is not well understood.

An incidental result of interest obtained by the author is the accuracy of the equivalence principle (implied by Eötvös' results) for various kinds of energy: electron rest-mass—1 in 10^4, atomic binding energy—1 in 400, electrostatic energy nucleus—2 in 10^6, nuclear binding energy—1 in 10^5.

The author's final conclusion is that, although the active gravitational masses of antiparticles remain unknown, the gravitational separation of stable aggregates of matter and antimatter on a cosmological scale cannot be achieved.

D. W. Sciama (London)

1882:

Lichnerowicz, André. Ondes électromagnétiques et ondes gravitationnelles en relativité générale. Cahiers de Phys. **12** (1958), 287–296.

The author considers the nature of permissible discontinuities in the electromagnetic field tensor and in the Riemann tensor, $R_{\alpha\beta,\lambda\mu}$. From the comparison, he proposes that in general relativity a state of pure total (i.e. electromagnetic and gravitational) radiation should fulfil the condition that there exist a vector l_α such that

$$l_\gamma R_{\alpha\beta,\lambda\mu}+l_\alpha R_{\beta\gamma,\lambda\mu}+l_\beta R_{\gamma\alpha,\lambda\mu}=0,$$
$$l^\alpha R_{\alpha\beta,\lambda\mu}=0.$$

If $R_{\alpha\beta,\lambda\mu}\neq 0$, l_α is necessarily null, and the Ricci tensor has the form $R_{\alpha\beta}=\tau l_\alpha l_\beta$. If $R_{\alpha\beta}=0$ the metric corresponds to pure gravitational radiation. It is shown that the metric of Bondi [Nature **179** (1957), 1072–1073 (letter to the editor)] corresponds to pure gravitational radiation in this sense.

Other topics briefly discussed are the motion of test particles in radiation fields, and the definition of such fields in the presence of sources.

W. B. Bonnor (London)

1883:

Papapetrou, A. Über zeitabhängige Lösungen der Feldgleichungen der allgemeinen Relativitätstheorie. Ann. Physik (7) **2** (1958), 87–96.

This paper is a continuation of previous work [same Ann. (6) **20** (1957), 399–411; (7) **1** (1958), 186–197; MR **19**, 1020; **20** #5675] in which the results concerning periodic gravitational fields are generalized for the case of an arbitrary dependence on time. It is shown with the help of the approximation procedure that the field becomes Euclidian at infinity only if the gravitational and electromagnetic field for $t=-\infty$ and $t=+\infty$ does not depend on time.

L. Infeld (Warsaw)

1884:

Bonnor, W. B. The mechanics of general relativity. Proc. Roy. Soc. London Ser. A **251** (1959), 55–65.

It is shown how to obtain, within the general theory of relativity, equations of motion for two oscillating masses at the ends of a spring of given law of force. The method of Einstein, Infeld and Hoffmann is used, and the force in the spring is represented by a stress singularity. The detailed calculations are taken to the Newtonian order.

D. W. Sciama (London)

1885:

Marder, L. Gravitational waves in general relativity. I. Cylindrical waves. Proc. Roy. Soc. London. Ser. A **244** (1958), 524–537.

The paper deals with the cylindrical gravitational field generated by an infinitely long material cylinder. First the field of a static cylinder is investigated both inside and outside of the cylinder. It is found that the latter can be characterized by two constants. Then the emission of a pulse wave is considered, and it is found that the result of the process is to bring about a permanent change in one of the constants characterizing the static field of the cylinder. An argument is presented to show that the pulse wave carries energy. A further calculation involving an assumed model for the cylinder as a wave source shows that the inertial mass of the latter decreases when a pulse wave is emitted. The case of periodic waves and the difficulties connected with them are also discussed.

N. Rosen (Haifa)

1886:

Marder, L. Gravitational waves in general relativity. II. The reflexion of cylindrical waves. Proc. Roy. Soc. London. Ser. A **246** (1958), 133–143.

"The reflexion of outward travelling cylindrical gravitational waves by a thin cylindrical shell of matter is discussed. The shell is regarded as a surface distribution, and it is shown that if it is perfectly rigid, total reflexion occurs. Quasi-periodic incident waves are found to lead to a completely periodic standing wave state between the reflector and a central wave source, the field outside the system remaining static. In the treatment of standing waves the components of the stress-energy tensor in the shell are determined and the construction for a model source is given." (Author's summary)

D. W. Sciama (London)

1887:

Jankiewicz, C. Über die Bewegung der Feldsingularitäten in kovarianten Feldtheorien. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 771–774.

The author claims to have shown that in every classical theory formulated in accordance with General Relativity Theory, the equations of motion of point singularities follow from the field equations. *L. Infeld* (Warsaw)

1888:

Blankfield, Judith; and McVittie, G. C. Einstein's equations and classical hydrodynamics. Arch. Rational Mech. Anal. **2** (1958/59), 337–354.

This paper is closely connected with that of McVittie [*General relativity and cosmology*, Wiley, New York, 1956; MR **19**, 370] in which it was shown that assuming an orthogonal metrical tensor of a somewhat more general form than that commonly used and an energy tensor appropriate to a perfect fluid, the equations of motion and of continuity of classical hydrodynamics can be developed. The chief aim of the present investigation is to put the approximation procedure on a sound basis by introducing two parameters—that involving the constant of gravitation and that involving the velocity of light. The approximation procedure by Einstein and Infeld corresponds only to the second choice. *L. Infeld* (Warsaw)

1889:

Cattaneo, C. General relativity: relative standard mass, momentum, energy and gravitational field in a general system of reference. Nuovo Cimento (10) **10** (1958), 318–337. (Italian summary)

The author seeks to show that the equations of motion of a free test particle referred to an arbitrarily chosen system of reference can quite generally be written in a form which can be read as "Time derivative of momentum = mass × gravitational field". The work seems to be largely formal in nature, and its meaning from an operationalist point of view is not easily assessed.

H. A. Buchdahl (Princeton, N.J.)

1890a:

Mishra, R. S. Einstein's connections. I. Tensor (N.S.) **9** (1959), 8–43.

1890b:

Mishra, R. S. Einstein's connections. II. Non-degenerate case. J. Math. Mech. **7** (1958), 867–892.

(The author uses the symbolism of the reviewer's book *Geometry of Einstein's unified field theory*, Noordhoff, Groningen, 1957 [MR **20** #5067]. The page numbers in parenthesis refer to this book.) The connection $\Gamma_{\lambda\mu}{}^{\nu}$ of Einstein's latest unified field theory is defined in terms of 64 equations (p. 50)

$$D_{\omega}g_{\lambda\mu} = 2S_{\omega\mu}{}^{\alpha}g_{\lambda\alpha}. \tag{1}$$

The solution must be of the form (p. 52)

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \left\{ {\nu \atop \lambda\ \mu} \right\} + S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu}, \tag{2a}$$

where

$$U^{\nu}{}_{\lambda\mu} \stackrel{\text{def}}{=} 2h^{\nu\alpha}S_{\alpha(\lambda}{}^{\beta}k_{\mu)\beta}. \tag{2b}$$

The first part of the first paper deals with the solution of (1) by Mme. Tonnelat [*La théorie du champ unifié d'Einstein et quelques-uns de ses développements* Gauthier-Villars, Paris, 1955; MR **17**, 907]. Mme. Tonnelat's claim that her solution is valid for all classes (for $\mathfrak{k} \neq 0$, $\mathfrak{k} = 0$) is based on the following erroneous assertion: The left hand term of

$$2(\mathfrak{k})^{1/2} \overset{-1}{k}{}^{\lambda\nu} = \mathfrak{E}^{\omega\mu\lambda\nu}k_{\omega\mu}$$

could be replaced by the right hand term even if $\mathfrak{k} = 0$. The author displays first some amusing consequences of this assertion and then shows the exact spot in Mme. Tonnelat's method where her procedure fails for $\mathfrak{k} = 0$. {In the first counterexample p. 12 one should read "Assuming $\mathfrak{k}^{1/2} \neq 0 \cdots$". The last term in (2.20) p. 14 should read $\partial\rho \ln k$.} Mme. Tonnelat also exhibits necessary and sufficient conditions for the existence of a solution. The author points out that these conditions are also satisfied for certain cases of index of inertia 4 which do not even exist. [Cf. also the erroneous statements in Kichenassamy, C. R. Acad. Sci. Paris **244** (1957), 168–170, 2007–2009; MR **18**, 704; **19**, 226.] In the second part of the first paper and in the second paper the author develops a method which yields the solution for all (four) classes and all indices of inertia: The tensors $S_{\omega\mu\nu}$ and $U_{\nu\omega\mu}$ are related by (p. 53)

$$2S_{\omega\mu\nu} = K_{\omega\mu\nu} - 4U_{\alpha\nu[\mu}k_{\omega]}{}^{\alpha}, \tag{E_0}$$

while for the first class (and $D \neq 0$)

$$^{(4)}k_{\lambda}{}^{\nu} + 2K^{(2)}k_{\lambda}{}^{\nu} + k\delta_{\lambda}{}^{\nu} = 0. \tag{3}$$

The substitution from (2b) into E_0 yields an equation E_1, the left hand term of which is again $2S_{\omega\mu\nu}$, while the right hand term contains quadratic products of $k_{\lambda}{}^{\nu}$. Substituting into the right hand term of E_1 from the left hand term of the same equation we obtain a new equation E_2, the left hand term of which is $2S_{\omega\mu\nu}$, while the right hand term contains products of degree p of $k_{\lambda}{}^{\nu}$ (pp. 2, 4). Iterating this procedure and taking into account (3), one obtains a finite set of different equations. Expressing them in the non-holonomic frame of eigenvectors of $k_{\lambda}{}^{\nu}$ (p. 19) one obtains a relatively simple set of equations which yield $S_{\omega\mu\nu}$ (and therefore $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$). The same methods hold also for the remaining three classes except that (3) acquires different forms for these classes.

{This is a short description of the main results of these two papers. It has to be stressed that this description is

by no means an exhaustive one. There are many interesting side results which were not even mentioned in this review.} *V. Hlavatý* (Bloomington, Ind.)

1891:

Mishra, R. S. Einstein's connections. III. Degenerate cases of second class. Nuovo Cimento (10) **10** (1958), 965–984. (Italian summary)

This is a continuation of the previous two papers with the same title (for the symbolism and the numbers of equations see above reviews of these papers). The reviewer found necessary and sufficient conditions for the existence of at least one solution of (1). In the degenerate cases $g(g-2)=0$ for the index of inertia 2 and exhibited all possible solutions (p. 114–123). The author of the paper under review applies his method of repeated substitution to the case $g(g-2)=0$ for all indices of inertia, finds necessary and sufficient conditions for the existence of at least one solution and displays explicitly all possible solutions. *V. Hlavatý* (Bloomington, Ind.)

ASTRONOMY

See also 1806, 1813.

1892:

Myachin, V. F. On the estimation of errors by numerical integration of the equations of celestial mechanics. Byull. Inst. Teoret. Astr. 7 (1959), 257–280. (Russian. English summary)

Results are presented of an application of a general theory for estimation of numerical integration errors to the differential equations of undisturbed motion of celestial bodies. The theory is mostly based on that proposed by S. M. Lozinsky but with regard to the random character of rounding errors. The theory confirms qualitatively Brouwer's well-known affirmation that after a sufficiently large number k of numerical integration steps the error in the coordinates of elliptical motion increases like $k^{3/2}$. *Author's summary*

1893:

Sočilina, A. S. On accumulation of errors in numerical integration in some problems of celestial mechanics. Byull. Inst. Teoret. Astr. 7 (1959), 281–286. (Russian. English summary)

Results of the application of the estimation method of errors in numerical integration [see previous review] to numerical examples are given. *Author's summary*

1894:

Amelin, V. M. The methods of making use of the moon for geodetic purposes. Byull. Inst. Teoret. Astr. 7 (1958), 19–42. (Russian. English summary)

The methods of making use of results of occultations, eclipses and special photographic observations of the moon for geodetic purposes are discussed. Some numerical examples are given. *Author's summary*

1895:

Makarova, E. N. On the simultaneous determination of systematic errors of stellar catalogues and of the masses of planets from observations of asteroids. Byull. Inst. Teoret. Astr. 7 (1958), 1–18. (Russian. English summary)

The classical Lagrangean equations of variations of elements in vectorial form are derived and compared with the systems of Musen and Herrick. Perturbations of Pallas by Jupiter from 1955 to 1965 are computed by Lagrange's and Musen's methods. The conditions for a simultaneous determination of systematic errors of stellar catalogues and the mass of Jupiter, based on observations of Pallas in this period, are discussed. It is shown that the introduction of a new unknown quantity (the correction to Jupiter's mass) does not affect the weights of constants of the catalogue. *Author's summary*

1896:

Zhevakin, S. A. The evaluation of non-adiabatic stellar pulsations by use of a discrete model. Astr. Ž. **36** (1959), 269–282. (Russian. English summary)

"The methods of evaluating radial non-adiabatic stellar pulsations, proposed by Woltjer and Rosseland, are considered in brief. It is pointed out that these methods require much more laborious calculations than the discrete model method.

A method for calculating radial non-adiabatic pulsations by use of a discrete model of a stellar envelope is elaborated. The necessity of taking into account the spheroidality of the stellar envelope is shown. The conditions of applicability of treatment by a discrete model are formulated.

A discrete multilayer spherical model of a pulsating stellar envelope is constructed. The equations of motion of this model are deduced and linearized, assuming radiative energy transfer." (Author's summary)

R. M. Evan-Iwanowski (Syracuse, N.Y.)

1897:

Bažienow, G. M. Zwei praktische Schemas zur Berechnung von Störungen erster Ordnung der Bahnelemente eines Körpers von geringer Masse im räumlichen restringierten elliptischen Dreikörperproblem. Byull. Inst. Teoret. Astr. 7 (1958), 43–71. (Russian. German summary)

In der vorliegenden Arbeit werden zwei praktische Schemas zur Berechnung von absoluten Störungen erster Ordnung der Bahnelemente eines Körpers mit verschwindend kleiner Masse im räumlichen restringierten elliptischen Dreikörperproblem geboten.

Zunächst werden die Grundgleichungen abgeleitet, mittelst derer die gesuchten Störungen bestimmt werden. Dann werden die auf der Theorie der besten Annäherungen von Tschebyscheff gegründeten Methoden der Reihenentwicklung von für die Auflösung der gestellten Aufgabe benötigten Grössen betrachtet und Rechenschemas für eine effektive Ableitung der Reihenkoeffizienten und für die Integration dieser Reihen angeführt.

In dem einem Schema wird die mittlere Anomalie des kleinen Körpers als Integrationsargument angenommen, in dem anderen seine exzentrische Anomalie.

Die beiden Schemas sind ungefähr gleichwertig. Die im ersten Schema vorkommenden Multiplikationen trigonometrischer Reihen von zwei Argumenten mit trigono-

metrischen Reihen von einem Argument werden im zweiten Schema durch einfachere Multiplikationen trigonometrischer Reihen von zwei Argumenten mit trigonometrischen Polynomen von einem Argument ersetzt, jedoch kommen im zweiten Schema mühevolle Operationen vor, bedingt durch die Einführung neuer Variabeln in drei Reihen.

Im Anhang werden Hilfstafeln und ein Rechnungsbeispiel nach dem ersten Schema für die Störungen der Bahnelemente des kleinen Planeten Sita (244) angeführt.

Zusammenfassung des Autors

1898:

Šmakova, M. Ya.; and Sočilina, A. S. An approximate method of determining of a circular orbit of an asteroid. Byull. Inst. Teoret. Astr. **7** (1958), 72–75. (Russian. English summary)

A simple method of determining the radius of a circular orbit can be obtained if we replace the condition of a circular orbit $v_2-v_1=ka^{-3/2}(t_2-t_1)$ by an approximate equality $\gamma=ka^{-3/2}(t_2-t_1)$, where γ is the known angle between the geocentric directions. The value of the radius thus obtained is a fair first approximation or in most cases its final value. *Authors' summary*

1899:

Simon, René. Radial oscillations of the generalized Roche model. Astrophys. J. **127** (1958), 428–435.

The radial adiabatic oscillations of the generalized Roche model (a self-gravitating configuration consisting of a homogeneous compressible core surrounded by a finite atmosphere in which the density falls off with the square of central distance) are investigated in this paper in the acoustic approximation. It is shown that such a model is capable of harmonic oscillation as a whole for any fractional dimensions of the core—in generalization of the results previously derived by Kopal [Astrophys. J. **111** (1950), 395–407] in which the density as well as pressure at the base of the atmosphere was considered negligible. If these are retained as finite, the characteristic frequencies of free oscillations are shown by Simon to be proportional to the square-root of the density of the core, and to be dependent on the fractional dimensions of this core as well as on the ratio of the masses of the core and the envelope, and on their specific heats. Application is also made to the case of a rigid nucleus, which may be of interest to the theory of planetary atmospheres. As far as the stars are concerned, however, any idealized model possessing a homogeneous nucleus of finite size is ruled out of existence by the stability requirements if any nuclear energy sources are present.

Z. Kopal (Manchester)

1900:

Carstoiu, John. Sur les champs magnétiques à force libre. C. R. Acad. Sci. Paris **248** (1959), 73–75.

In the Beltrami equation $\nabla\times\mathbf{H}=\alpha\mathbf{H}$ for force-free fields, the author considers the case where α is no longer a constant. *C. H. Papas* (Pasadena, Calif.)

GEOPHYSICS

See also 1921.

1901:

Strahov, V. N. On the theory of a two-dimensional problem of magnetic prospecting. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1959**, 244–253. (Russian)

In the interpretation of results of a magnetic survey the unknowns are the shape and depth of magnetized masses which cause the observed anomalies as well as the direction and intensity of their magnetization. The author considers the so-called two-dimensional case when it is known (or postulated) that the magnetized body C has a cylindrical shape (with an unknown normal section), so that the anomaly is characterized by magnetic profiles. In this case the usual and well known approach to the problem of interpretation was to form the complex magnetic potential $W(z)$, so that $-W'(z)$ represents the complex anomaly $H+iZ$ and then apply the theory of analytic functions. The author introduces a generalized potential $W_*(z)=(p+iq)W(z)$, where p and q are any two constants and constructs a general theory so that all the classical problems appear as particular cases of one and the same problem for various choices of p and q.

Naturally it does not change or simplify the real problem of interpretation, but from the point of view of numerical computations with the aid of electronic computing equipment it may prove important as a basis for a general program. *E. Kogbetliantz* (New York, N.Y.)

OPERATIONS RESEARCH, ECONOMETRICS, GAMES

See also 1315, 1677, 1693, 1758.

1902:

Arrow, Kenneth J.; Block, H. D.; and Hurwicz, Leonid. On the stability of the competitive equilibrium. II. Econometrica **27** (1959), 82–109.

This is the second of two connected articles. The first, by Arrow and Hurwicz, appeared in Econometrica **26** (1958), 522–552 [MR **21** #629]. In the present article proofs of stability have been extended to more general types of dynamic "adjustment equations". Of particular interest is the study of stability of absolute prices as compared to the stability of relative prices.

T. Haavelmo (Oslo)

1903:

Brown, Murray. The structure of stochastic difference equation models. Econometrica **27** (1959), 116–120.

Comments, in terms of a simple example, on the fact that in a system of homogeneous linear difference equations with constant coefficients, the amplitudes of the several variables are determined jointly by the matrix of coefficients and the initial conditions.

R. Solow (Cambridge, Mass.)

1904:

Newell, Gordon F. The effect of left turns on the capacity of a traffic intersection. Quart. Appl. Math. **17** (1959), 67–76.

"A model is proposed for the traffic flow through a fixed-cycle traffic signal on a narrow two-lane highway. The average flows in the two opposing lanes are computed when the queues are arbitrarily long, assuming that left turns occur with fixed probabilities in the two lanes. It is

found that the existence of left turns tends to favor short cycles of the light, and, under certain conditions, the competition between this effect and the advantages of the long cycle gives rise to an optimal cycle time at which the average flows have a maximum value." (From the author's summary) *E. Reich* (Minneapolis, Minn.)

1905:

Kometani, Eiji; and Sasaki, Tsuna. On the stability of traffic flow (report-I). J. Operations Res. Soc. Japan **2** (1958), 11–26.

It is assumed here that one car follows another in such a way that its velocity at time t is some linear function of the spacing between the two cars measured at time $t-T$ with T the driver's reaction time. This is the same model as one of the car following theories investigated by Chandler, Herman and Montroll [Operations Res. **6** (1958), 165–184; MR **20** #770] and by Herman, Montroll, Potts and Rothery [see review below], in particular the one which the latter authors found to be experimentally most satisfactory. The mathematical analysis and conclusions in the present paper are nearly identical with those in the above mentioned papers although this paper was written independently and at nearly the same time. *G. Newell* (Stockholm)

1906:

Herman, Robert; Montroll, Elliott W.; Potts, Renfrey B.; and Rothery, Richard W. Traffic dynamics: analysis of stability in car following. Operations Res. **7** (1959), 86–106.

This is a sequel to the paper by Chandler, Herman and Montroll [Operations Res. **6** (1958), 165–184; MR **20** #770] dealing with car following models in which the trajectory of the jth car of a sequence is related to that of the $(j-1)$th car through a linear differential difference equation. By investigating the solutions of these equations with Laplace transform techniques, the authors find that if the product C of the coupling constant between cars and the time lag in response is sufficiently large, the motion is unstable in the sense that nearly any disturbance will increase exponentially in time. There is a second range of C in which the transients of a disturbance have the form of a damped oscillation and finally a third range of small values for C in which the transients are pure exponentials.

The authors also give some further comments on the type of stability discussed in their previous paper, the decrease in amplitude of a disturbance as it propagates from one car to the next in a long sequence, and they describe some effects of couplings between the jth and $(j-2)$th or j and $(j+1)$th cars. Finally, some discussion and experimental data show certain statistical properties of the acceleration of a car considered as a time series. *G. Newell* (Stockholm)

1907:

Berger, Gottfried. Eine besondere Deutung der Äquivalenzgleichungen und ihre Anwendung auf die Lebensversicherung erhöhter Risiken. Bl. Deutsch. Ges. Versicherungsmath. **4** (1959), 141–172.

Der Verfasser gibt eine spezielle Deutung der zu einer gemischten Versicherung gehörigen Aequivalenzgleichungen. Mit einer Approximationsmethode, die auf der Ansammlungstheorie von Cantelli und einer speziellen, von C. Boehm gefundenen Prämienformel beruht, werden Untersuchungen betreffend rechnerische Konsequenzen der Konstanten sowie der degressiven multiplikativen Uebersterblichkeit dargestellt. Theoretische und numerische Abschätzungen beleuchten die Güte der Approximation, sowie Vergleichungen mit den entsprechenden Untersuchungen und Beispielen von H. Jecklin. *W. Saxer* (Zürich)

1908:

★Beckmann, Martin J. Lineare Planungsrechnung. Linear programming, Bd. 1. Mit einem Geleitwort von E. Gutenberg. Wirtschaftswissenschaft der Gegenwart, I. Planungsforschung. Operations Research. Fachverlag für Wirtschaftstheorie und Ökonometrie, Ludwigshafen am Rhein, 1959. x+118 pp. Brosch.: DM 9.80. Ganzleinen: DM 13.80.

Linear programming is discussed within the framework of neoclassical economic theory with emphasis on general principles rather than numerical techniques. Chapter I treats linear production models, which serve to introduce basic concepts of linear programming. Chapter II illustrates the general principles by examples from activity analysis. The principles are discussed in a heuristic manner, formal proofs being relegated to Chapter V, which also contains a three-page sketch of the simplex method. Chapter III is concerned with other applications of linear programming (e.g. the transportation, diet, production smoothing, and assignment problems) and Chapter IV with generalizations (e.g. Pareto maxima, convex programming, and stochastic linear programming). As the book is meant for economists rather than mathematicians, formulas are used sparingly, verbal statements being preferred, wherever they can be phrased concisely. The exposition is clear and easy to follow. *W. Prager* (Providence, R.I.)

1909:

★Gurk, Herbert M. Five-person, constant-sum, extreme games. Contributions to the theory of games, Vol. IV, pp. 179–188. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N. J., 1959. xi+453 pp. $6.00.

A K-chain for a game is a chain $T_1, \cdots, T_{2K}$ of subsets of players such that (i) $T_1=T_{2K}$, (ii) $T_j \cap T_{j+1}=0$, (iii) $v(T_j)+v(T_{j+1})=1$. The author shows that a necessary and sufficient condition that a constant-sum five-person game be extreme is that $v(S)\neq 0, 1$ (in the $0-1$ normalization) implies the existence of a K-chain starting with S. For n-person games with $n \geq 6$ the condition is sufficient but not necessary. This condition enables one to enumerate all extreme five-person games. There are 294 extreme five-person games of which 76 are simple. The author provides solutions for all extreme five-person games. *E. D. Nering* (Boulder, Colo.)

1910:

★Luce, R. Duncan. A note on the article "Some experimental n-person games". Contributions to the theory of games, Vol. IV, pp. 279–285. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N. J., 1959. xi+453 pp. $6.00.

The author analyses an experiment performed by Kalisch et al. [R. M. Thrall, ed., *Decision processes*, Wiley, New York, 1954; MR **16**, 605; pp. 301–327] with regard

to his theory of ψ-stability [Luce, Ann. of Math. (2) **59** (1954), 357–366; MR **15**, 975]. A subjective description given by Kalisch et al. of the behavior of their subjects leads to a function ψ for which the predictions of the theory of ψ-stability and the data of the experiment are remarkably consistent. The author also suggests that the data are not as inconsistent with the theory of strategic equivalence as Kalisch et al. indicate.

E. D. Nering (Boulder, Colo.)

1911:

★Kalisch, G. K.; and Nering, E. D. Countably infinitely many person games. Contributions to the theory of games, Vol. IV, pp. 43–45. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N. J., 1959. xi + 453 pp. $6.00.

The authors consider a zero sum game with a countable set of players, in normalized characteristic function form. Imputations are as usual, with the convergence to zero required to be absolute. They display a class of such games having no solutions and draw the conclusion that the ordering relation of domination alone is insufficient for proving existence of solutions for all finite n-person games.

J. H. Blau (Yellow Springs, Ohio)

1912:

Isbell, John R. On the enumeration of majority games. Math. Tables Aids Comput. **13** (1959), 21–28.

Majority games are simple games which can be defined by a set of numerical weights, w_i, one for each player. The winning coalitions are those which have more than half the total weight. The author finds a necessary and sufficient condition for the existence of an $(n+1)$-person majority game in terms of pairs of n-person majority games. Based on this theorem a combinatorial method is given for generating all such games. Although enumeration of all these games cannot be accomplished in general for all n, rough upper and lower bounds of 2^{2^n} and 2^n respectively are determined.

The author also discusses efficient majority games, whose weights are integers such that winning sets have total weights exceeding those of losing sets by just one. Several results are obtained concerning the generation of efficient sequences of weights. In particular a computational procedure is described for enumerating these games also. Bounds of 2^{n-4} and $(n-1)!$ are established for such n-person games.

H. M. Gurk (Princeton, N.J.)

BIOLOGY AND SOCIOLOGY

1913:

★Huron, Roger; et Ruffié, Jacques. Les méthodes en génétique général et en génétique humaine. Préface de Albert Vandel. Masson et Cie, Paris, 1959. viii + 556 pp. 8200 francs.

The first part of this book deals with general genetics, genes, linkage, mutation, chromosomal abberations, etc., more from the biological than the statistical point of view. In the second and longer part, devoted to human genetics, the emphasis is much more on the statistical aspect. The fundamental principles of statistics are explained in detail (probability, the binomial and Poisson distributions, significance tests, t, analysis of variance, regression, correlation, etc.) but the algebra is kept at a reasonably elementary level. Questions of specific genetic interest discussed include Mendelian ratios, testing and estimation of linkage, gene frequencies, selection, cousin marriage: here there are gathered together statistical techniques which otherwise are mainly scattered in original papers.

C. A. B. Smith (London)

INFORMATION AND COMMUNICATION THEORY

See also 1530, 1683a-b, 1684, 1685.

1914:

Brillouin, L. Inevitable experimental errors, determinism, and information theory. Information and Control **2** (1959), 45–63.

Some of the comments in this paper are well, and some are ill, taken. The geometric examples (due to Borel) of Liouville's theorem and its connection with uncertainty are very helpful; on the whole, there are too many slogans. The author contrasts the classical view, that experimental errors are accidental, with the modern admission that experimental disturbance is not negligible. The former view, he claims, is that of mathematicians discussing the axioms of geometry. This is misleading, since classical and modern physics alike consist of an interpreted mathematical framework. He seems to forget that "determinism" and "uncertainty" are essentially properties of these differing mathematical frames, saying that determinism is a metaphysical creed and Heisenberg's principle a physical law. In a sense, both notions are outgrowths of prowess in analysis, the first from differential equations, the second from square-integrable functions and their Fourier transforms, or more generally, from canonical operators.

V. E. Beneš (Murray Hill, N.J.)

1915:

Dobrušin, R. L. Transmission of information in channels with feedback. Teor. Veroyatnost. i Primenen. **3** (1958), 395–412. (Russian. English summary)

Consider a discrete channel Γ which comprises a forward channel Γ_d and a feedback channel Γ_f, and let $\Gamma^{(n)}$ be its extension of length n. Let $\mathscr{E}^{(n)}$ and $\bar{\mathscr{E}}^{(n)}$ denote, respectively, the sets of all input and output sequences of length n, and let $\mathscr{F}$ and $\bar{\mathscr{F}}$ denote the corresponding sets of input and output messages. In the process of transmission, a message $\eta\in\mathscr{F}$ is encoded into a sequence $\xi\in\mathscr{E}^{(n)}$ which is received as a sequence $\bar{\xi}\in\bar{\mathscr{E}}^{(n)}$, which is then decoded as a message $\bar{\eta}\in\bar{\mathscr{F}}$. Thus η and $\bar{\eta}$ represent the input and output of $\Gamma^{(n)}$, and ξ and $\bar{\xi}$ are those of Γ_d. In the case of a channel with feedback, the encoding of η into ξ is affected by the knowledge of the received sequence $\bar{\xi}$. The capacity of Γ_f is assumed to be arbitrarily large.

The main results of the paper are contained in two theorems concerning channels with feedback. Theorem (a): If Γ_d is memoryless, then $I(\eta,\bar{\eta})\leq C^{(n)}$, where $I(\eta,\bar{\eta})$ is the information conveyed by $\bar{\eta}$ about η, and $C^{(n)}$ is the capacity of $\Gamma_d^{(n)}$. The author shows by a counter example that $I(\eta,\bar{\eta})\leq C^{(n)}$ does not hold in general for channels with memory. As a special case of channels with memory, the author considers so-called randomized memoryless

channels, that is, channels in which the transition probability matrix $P=[p_{ij}(\alpha)]$ depends on a random variable α, with each fixed value of α giving rise to a memoryless channel. Assuming that α ranges over a finite set $a_1, a_2, \cdots, a_s$ with respective probabilities $q(a_1), \cdots, q(a_s)$, theorem (a) can be generalized to theorem (b): If Γ_d is a randomized memoryless channel, then $I(\eta, \bar{\eta}) \leq n\bar{K}$, where $\bar{K}=\sum_\mu q(a_\mu)D(a_\mu)$ and $D(a_\mu)$ is the channel capacity of Γ_d with $\alpha=a_\mu$. *L. A. Zadeh* (Berkeley, Calif.)

1916:

Blyth, Colin R. Note on estimating information. Ann. Math. Statist. **30** (1959), 71–79.

Let Y be a random variable such that $\Pr(Y=\alpha_i)=p_i$, $i=1, 2, \cdots, k$. The author considers estimates of the Shannon-Wiener measure of information of Y, $H=H(p_1, p_2, \cdots, p_k)=-C\sum_{i=1}^k p_i \log_2 p_i$ where $C=\log_2 e=1.442695$, based on independent observations of Y. The maximum likelihood estimate has been studied by Miller and Madow [U.S. Air Force, Cambridge Res. Center, Tech. Rep. 54–75, 1954]. Since there is no unbiased estimate, the author considers best estimates with low bias and obtains the bias and variance. These estimates are extended to the case where the number of possible values of the random variable is unknown. The estimates are compared asymptotically with the maximum likelihood estimates. They are also compared with the minimax estimates (for squared error loss function) for a few special cases where these are easily found.

S. Kullback (Washington, D.C.)

1917:

Dobrušin, R. L. A simplified method of experimentally evaluating the entropy of a stationary sequence. Teor. Veroyatnost. i Primenen. **3** (1958), 462–464. (Russian. English summary)

Consider a sequence of independent random variables $\cdots, \xi_{-1}, \xi_0, \xi_1, \cdots$ ranging over a set of symbols $E_1, E_2, \cdots, E_m$ with respective probabilities

$$p_i = P\{\xi_j = E_i\} \quad (i = 1, 2, \cdots, m).$$

Let η be the time interval between successive occurrences of the same symbol. Then

$$P\{\eta > j|\xi_0 = E_i\} = (1-p_i)^j$$

or approximately

$$P\{\eta > x|\xi_0 = E_i\} \approx \exp(-xp_i)$$

for small p_i. From this it readily follows that the expectation of $\log \eta$ is approximately equal to $H-C$, where H is the entropy of the process and C is Euler's constant. The author uses this observation as a basis for estimating H. The accuracy of the estimate depends primarily on the goodness of the approximation

$$P\{\eta > x|\xi_0 = E_i\} \approx \exp(-xp_i).$$

The author states that there is some evidence that for small p_i this approximation may be valid for a fairly broad class of sequences of dependent variables.

L. A. Zadeh (Berkeley, Calif.)

1918:

Peterson, W. W.; and Rabin, M. O. On codes for checking logical operations. IBM J. Res. Develop. **3** (1959), 163–168.

This paper considers digit by digit logical operations and the problem of encoding information for such equipment so that the results are checked. No use is made of the details of the actual circuits, and the general conclusions are not encouraging in that they show that there is nothing simpler (except for trivial cases) than duplication. *R. W. Hamming* (Murray Hill, N.J.)

1919:

Blum, Marvin. On exponential digital filters. J. Assoc. Comput. Mach. **6** (1959), 283–304.

"This paper derives the weighting sequence of a linear digital filter whose output is an estimate of the predicted values of the derivatives of the input. The input functions considered are arbitrary linear combinations of $n+1$ known functions, plus a random stationary signal and a random stationary noise component. The filter differs from previously considered minimum variance optimum filters in that the primary consideration here is the computational ease with which one can obtain the final solution. An optimization in the minimum variance sense is obtained as a secondary consideration in order to provide some control of the mean square output error. The exponential filter has its simplest form for the class of nonrandom input functions (h_n) which are the complete solutions of a set of homogeneous linear difference equations of order n with constant coefficients. For this class the input and output are related by a time invariant recursion formula. The output contains a bias error which can be made to approach zero exponentially as the mean square error increases monotonically to a limit with increasing time.

A modification of the exponential filter is considered such that the bias error is zero. The solution then involves a recursion formula with time varying coefficients." (Author's abstract) *R. W. Hamming* (Murray Hill, N.J.)

1920:

Shannon, Claude E. Probability of error for optimal codes in a Gaussian channel. Bell System Tech. J. **38** (1959), 611–656.

This paper gives asymptotic upper and lower bounds for errors in a continuous channel with a gaussian noise for codes subject to an average power limitation. These bounds are close to each other for signalling rates near the maximum capacity and near zero, and diverge between.

"The geometrical approach we use is akin to that previously used by the author but carried here to a numerical conclusion. ... It might be said that the algebra involved is in several places unusually tedious."

R. W. Hamming (Murray Hill, N.J.)

1921:

Dady, Guy. La théorie de l'information dans les problèmes météorologiques. J. Sci. Météorol. **10** (1958), 121–131. (English and Spanish summaries)

Information theory finds a natural application in meteorology. Its science of coding has application not only to the transmission of observations but also to the storing of large volumes of observations. The idea of quantity of information allows an estimate to be made of

the quality of a forecast. In this regard the author recommends as a measure of information about an event y knowing the event x,

$$I(y/x) = \frac{H(x)+H(y)-H(x, y)}{H(x)},$$

where $H = -\sum p \log p$ is the entropy.

H. H. Campaigne (Jessup, Md.)

SERVOMECHANISMS AND CONTROL

1922:

Aĭzerman, M. A.; and Gantmaher, F. R. Determination of periodic solutions in systems with piecewise linear characteristics comprising segments parallel to two given lines. Avtomat. i Telemeh. **18** (1957), 97–110. (Russian. English summary)

This is an extension of earlier work by the same authors [Prikl. Mat. Meh. **19** (1955), 222–224; MR **17**, 487] in which they had observed that the usual method of determining periodic solutions in relay systems can be generalized to relays whose characteristics comprise parallel but not necessarily horizontal lines. Here they consider systems characterized by the differential equations

$$\dot{x}_j = \sum_{k=1}^{n} a_{jk}x_k + \lambda_j f(x_1) + F_j(t) \quad (j = 1, 2, \cdots, n),$$

in which the a_{jk} and λ_j are constants, $f(x_1)$ represents an element whose characteristic comprises two straight but not necessarily parallel lines, and the $F_j(t)$ are periodic functions of time with a common period T. By employing a linear coordinate transformation the two straight lines defining $f(x_1)$ can be transformed into the coordinate axes. Using this simplifying device and assuming that the transitions from one axis to another occur only at the beginning of the period, at the end of the period and at most once at an intermediate point, the authors obtain a set of simultaneous algebraic equations for the parameters involved in the assumed periodic solution. No criteria are given for ascertaining in advance whether solutions of the assumed form exist. *L. A. Zadeh* (Berkeley, Calif.)

1923:

Čžan, Sy-in. On a stability criterion for nonlinear control systems. Avtomat. i Telemeh. **20** (1959), 669–672. (Russian)

For nonlinear systems characterized by the canonical equations

$$\dot{x}_k = -\rho_k x_k + f(\sigma) \quad (k = 1, 2, \cdots, n+1),$$

$$\sigma = \sum_{k=1}^{n+1} \gamma_k x_k,$$

$$\dot{\sigma} = \sum_{k=1}^{n+1} \beta_k x_k - r' f(\sigma),$$

where the $\rho_k, \gamma_k, \beta_k$ and r' are constants, Letov [*Ustoĭčivost' nelineĭnyh reguliruemyh sistem*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955; MR **17**, 487] obtained a sufficient condition for stability of the trivial solution in the form (1) $F(x, y, R) > 0$, where x and y are functions of the β_k, ρ_k and r', and R is an arbitrary positive constant. The author shows that the envelope of the family of curves $F(x, y, R) = 0$ defines a region of stability which is the union of regions defined by (1) for each positive R. The elimination of R from (1) and $\partial F/\partial R = 0$ yields the corresponding criterion of stability

$$\left[2r' - \sum_{k=1}^{n+1} (\beta_k/\rho_k)\right]^2 > \left[\sum_{k=1}^{n+1} (\beta_k^2/\rho_k)\right]\left[\sum_{k=1}^{n+1} (1/\rho_k)\right].$$

L. A. Zadeh (Berkeley, Calif.)

1924:

Rose, Alan. The use of universal decision elements as flip-flops. Z. Math. Logik Grundlagen Math. **4** (1958), 169–174.

A universal decision element ϕ is a functor such that all functors of one and two arguments are definable in terms of ϕ and the logical constants 0, 1, in such a way that only one occurrence of ϕ is used. A "flip-flop" is a device with two inputs and one output; initially all three of these variables are equal to F; if one input changes to T, the output changes to T and remains there even if that input reverts to F; if now the other input becomes T, the output again becomes F, and stays so even if this second input reverts to F. Methods for constructing flip-flops out of universal decision elements are discussed.

V. E. Beneš (Murray Hill, N.J.)

1925:

Leagus, Dolores C.; Lee, C. Y.; and Mealy, George H. Verification of the logic structure of an experimental switching system on a digital computer. Bell System Tech. J. **38** (1959), 467–476.

"The verification problem is concerned with the construction on a computer of a logical program which satisfies all the design specifications prescribed for an experimental switching system and with the process of putting calls through the computer simulation to evaluate the system's logical structure." (Author's summary)

J. P. Roth (Yorktown Heights, N.Y.)

1926:

Rose, Alan. Applications of logical computers to the construction of electrical control tables for signalling frames. Z. Math. Logik Grundlagen Math. **4** (1958), 222–243. (1 insert)

This paper, descriptive in character, shows how problems of switching railway trains and selecting routes for many trains at once can be formalized in terms of the propositional calculus of two truth-values, and programmed for suitable computing machines.

V. E. Beneš (Murray Hill, N.J.)

MISCELLANEOUS

1927:

★**Mostowski, Andrzej; i Stark, Marceli. Elementy algebry wyższej. [Elements of higher algebra.]** Biblioteka matematyczna. Tom 16. Państwowe Wydawnictwo Naukowe, Warsaw, 1958. 368 pp.

This book has arisen from the first two parts of the

book *Higher algebra* by the same authors [*Algebra wyższa*, I, II, III, Państwowe Wydawnictwo Naukowe, Warsaw, 1953, 1954; MR **15**, 594; **16**, 104, 663]. It is a university text-book of Algebra for first year students of Mathematics. The Table of contents: I. Introduction; II. Combinatorics; III. Complex numbers; IV. Determinants; V. Vector spaces and linear equations; VI. Polynomials in one variable; VII. The rings of polynomials over the real and complex fields; VIII. Quadratic, cubic and biquadratic equations; IX. The ring of polynomials with integral coefficients, algebraic and transcendental numbers; X. Polynomials in several variables, symmetric functions; XI. The theory of elimination; XII. Quadratic forms. An appendix is added about special properties of matrices and determinants. The book is written clearly, with great stress on didactic principles of the presentation.

M. Fiedler (Prague)

1928: √

★**Irving, J.; and Mullineux, N. Mathematics in physics and engineering.** Pure and Applied Physics, Vol. 6. Academic Press, New York-London, 1959. xvii + 883 pp. $11.50.

As stated in the preface, the topics have been chosen so that the reader will be in a position to understand the mathematics which appears in current technological journals. Space is saved by not making any thorough going attempt to derive the equations of mathematical physics but rather to give methods for their solution. There are 13 chapters on the topics: partial d.e.'s, ordinary d.e.'s (Frobenius' and other methods), Bessel and Legendre functions, Laplace and other transforms, matrices, analytical methods in classical and wave mechanics, calculus of variations, complex variable theory and conformal transformations, calculus of residues, transform theory, numerical methods, integral equations. There is an appendix (pp. 772–855) with definitions and formulae from pure mathematics. Each chapter has numerous problems, with solutions.

1929: √ 3rd ed of 1912

★**Webster, Arthur Gordon. The dynamics of particles and of rigid, elastic, and fluid bodies: Being lectures on mathematical physics. 2nd ed.** Dover Publications, Inc., New York, 1959. xii + 588 pp. $2.35.

This edition is an unabridged and unaltered republication of the second edition of 1912 [Teubner, Leipzig].

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